Generalized Bell inequalities and frustrated spin systems

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We find a close correspondence between generalized Bell inequalities of a special kind and certain frustrated spin systems. For example, the Clauser-Horn-Shimony-Holt inequality corresponds to the frustrated square with the signs $+++-$ for the nearest neighbor interaction between the spins. Similarly, the Pearle-Braunstein-Cave inequality corresponds to a frustrated even ring with the corresponding signs $+\ldots+-$. Upon this correspondence, the violation of such inequalities by the entangled singlet state in quantum mechanics is equivalent to the spin system possessing a classical coplanar ground state, the energy of which is lower than the Ising ground state's energy. We propose a scheme which generates new inequalities and give further examples, the frustrated hexagon with additional diagonal bonds and the frustrated hypercubes in $n = 3, 4, 5$ dimensions. Surprisingly, the hypercube in $n = 4$ dimensions yields an inequality which is *not* violated by the singlet state. We extend the correspondence to other entangled states and XXZ-models of spin systems.

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I. INTRODUCTION

Bell's inequality, published more than four decades ago, has not ceased to invoke keen interest in the physics community. The title of the seminal paper of J. Bell [1] refers to the famous article of A. Einstein, B. Podolski, and N. Rosen [2] (EPR) who concluded that, according to their criteria, quantum theory (QT) is incomplete. Bell proved that the assumptions of EPR lead to an inequality for measurable correlations of spin measurements for two particles which is violated in QT, and, as later work showed, also in experiments. Thus certain kinds of hidden variable theories are empirically ruled out. Of course, the exact relation between Bell's assumptions and those of EPR has to be carefully examined. According to an analysis of L. E. Ballentine and J. P. Jarrett [3] the assumptions of Bell, simple locality and predictive completeness, are even weaker than the EPR assumptions. Hence, in the words of these authors, the *incompleteness* of QT *is, in some sense, a property of nature* [3].

There have been many proposals to generalize Bell's inequality. An important generalization is the Clauser-Horn-Shimony-Holt inequality [4] (CHSH) which is about a linear combination of the correlations of two pairs of measurements. A generalization to n pairs of measurements has been considered by Pearle [5] and later investigated by Braunstein and Cave [6]. Other work generalized Bell's inequality to an arbitrary number of measurements [7] or to more than two particles [8]. See the textbook of A. Peres [8] and literature quoted there for more details.

In this article we will point out a close correspondence between possible generalized Bell inequalities (GBI's) and certain frustrated *classical* spin systems Σ_N^{cl}. These terms will be explained in more detail below. It is important to distinguish the spin system Σ_N^{cl} from the *quantum* spin system Σ_2^{q} on which the EPR measurements are performed. The number N of spins in Σ_N^{cl} corresponds to the number of measurements considered in the context of GBI's. As a by-product of the correspondence we will obtain a procedure to generate new GBI's including a test whether these inequalities are violated in QT.

This paper is organized as follows: In section II we explain the basic idea of the correspondence using the example of the CHSH inequality. Then we will give the general definitions for the spin systems which give rise to a correspondence with GBI's. The GBI is violated by the singlet state if and only if the corresponding Heisenberg spin system has a classical ground state with a lower energy than the corresponding Ising ground state. These systems thus have necessarily non-collinear ground state configurations, i. e. coplanar or 3-dimensional ones, although in all examples considered in this article it is not necessary to consider 3-dimensional ground states. In section III we present some methods to calculate classical ground states and apply these to the construction procedure for GBI's in section IV. Section V is devoted to a couple of examples, including the frustrated $2n$-ring leading to the Pearle-Braunstein-Cave inequality, the frustrated hexagon and the frustrated hypercubes H_n. In all these examples it is possible to analytically calculate classical Heisenberg ground states and Ising ground states and to compare their ground state energies. With the exception of H_4 the Ising ground state energy is higher and hence we obtain GBI's violated in QT by the singlet state. More general entangled states are considered in section VI and are shown to lead to XXZ-models of spin systems. We close with a conclusion in section VII. The correspondence between GBI's and frustrated spin systems is summarized in table II.

II. INEQUALITIES AND SPIN SYSTEMS

In order to motivate the correspondence between generalized Bell inequalities (GBI's) and frustrated spin systems we consider the CHSH inequality [4], following [8].
Let a, b, c, d be four numbers which assume only the values ± 1. Then either $a + c = 0$ or $a - c = 0$ and hence $(a + c)b + (a - c)d$ assumes only the values ± 2, which yields the inequality

$$-2 \leq ab + ad + cb - cd \leq 2 \; . \tag{1}$$

Imagine that the numbers $a_i, b_i, c_i, d_i,\ \ i = 1, \ldots, N$ are the outcomes of four N-times repeated experiments and consider the mean values of the above products, i. e. the correlation,

$$\langle ab \rangle \equiv \frac{1}{N} \sum_{i=1}^{N} a_i \, b_i, \quad \text{etc.} \quad . \tag{2}$$

Then the inequality (1) also holds for the mean values, i. e.

$$-2 \leq \langle ab \rangle + \langle ad \rangle + \langle cb \rangle - \langle cd \rangle \leq 2 \; . \tag{3}$$

This is an equivalent form of the CHSH inequality [4] which belongs to the family of generalized Bell inequalities GBI. It bounds the classical correlations between four different measurements. We have formulated it without the assumption that the mean values converge against some expectation values for $N \to \infty$, following [8] since it seems to be more appropriate to consider this as part of the postulates which connect the CHSC inequality to measurements or to QT. These postulates are known to lead to a contradiction. The CHSC inequality alone is a mathematically valid statement about $N \times 4$-matrices with entries ± 1.
Now consider in QT a pair of particles with spin $s = \frac{1}{2}$ in its entangled singlet ($S = 0$) spin state

$$\phi = \frac{1}{\sqrt{2}} \left(| \uparrow\downarrow \rangle - | \downarrow\uparrow \rangle \right) \; . \tag{4}$$

Further consider measurements of the single particle spin in direction of the unit vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$. These are measurements of observables represented by the Hermitean operators

$$A = \vec{a} \cdot \vec{\sigma} = a_x \sigma_x + a_y \sigma_y + a_z \sigma_z \; , \tag{5}$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices and the observables B, C, D for the other directions $\vec{b}, \vec{c}, \vec{d}$ are analogously defined. It is possible to combine any two of these measurements and to measure, say, A at the left-hand particle and B at the other one. The two experimenters doing these measurements are traditionally called "Alice" and "Bob". According to the rules of QT, the combined measurement is represented by the tensor product operator $A \otimes B$. If a_i, b_i are the outcomes of N repetitions of this combined measurements, the mean values $\langle ab \rangle$ according to (2) converge towards the expectation value, which can be calculated by QT and depends on the state of the system. For the singlet state (4) the expectation value turns out to be

$$\langle AB \rangle \equiv \langle \phi | A \otimes B | \phi \rangle = -\vec{a} \cdot \vec{b} \; , \tag{6}$$

and analogously for $\langle AD \rangle, \langle CB \rangle$ and $\langle CD \rangle$. Hence the correlation term in the CHSH inequality (3) has the quantum theoretical counterpart

$$\langle AB \rangle + \langle AD \rangle + \langle CB \rangle - \langle CD \rangle = -\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{d} - \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{d} \; . \tag{7}$$

It is easily seen that the extreme values of (7) exceed the bounds of the CHSH inequality. The possible values of (7) are symmetric with respect to 0, since the substitution $\vec{a} \mapsto -\vec{a}, \vec{c} \mapsto -\vec{c}$ changes the overall sign in (7). Writing (7) in the form $-\vec{a} \cdot (\vec{b} + \vec{d}) - \vec{c} \cdot (\vec{b} - \vec{d})$ it is obvious that each term is minimal for the choice $\vec{a} \,||\, (\vec{b} + \vec{d})$ and $\vec{c} \,||\, (\vec{b} - \vec{d})$. Moreover, $|\vec{b} + \vec{d}| + |\vec{b} - \vec{d}|$ is maximal for $\sphericalangle(\vec{b}, \vec{d}) = 90°$. This is equivalent to the statement: The square has the maximal circumference among the rectangles with fixed length of their diagonals. Hence

$$-2\sqrt{2} \leq \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{d} + \vec{c} \cdot \vec{b} - \vec{c} \cdot \vec{d} \leq 2\sqrt{2} \; , \tag{8}$$

or

$$-2\sqrt{2} \leq \langle AB \rangle + \langle AD \rangle + \langle CB \rangle - \langle CD \rangle \leq 2\sqrt{2}\,, \tag{9}$$

and the lower bound is assumed for any configuration with $\sphericalangle(\vec{b},\vec{d}) = 90^\circ, \sphericalangle(\vec{c},\vec{d}) = 45^\circ, \sphericalangle(\vec{a},\vec{c}) = 90^\circ$, see figure 4. The upper bound is assumed similarly. Here and in what follows we always count the angles $\sphericalangle(\vec{a},\vec{b})$ between two unit vectors counter-clockwise, beginning with $\vec{a}$. Equation (9) expresses the bounds for quantum correlations between four possible measurements. Since the bounds are attained the CHSH inequality (3) is violated in QT for suitable measurements and entangled states.

The violation of the CHSH and similar inequalities, which are rigorously proven theorems, can only be understood in the sense that some of the assumptions leading to (3) must not hold in QT. Indeed, if $[A, C] \neq 0$ and $[B, D] \neq 0$, only one of the four possible combinations of measurements AB, AD, CB, CD can be performed as a joint measurement and hence only two of the four numbers a, b, c, d can be actually measured in one single experiment. If QT is right, it is thus not possible, by whatever means, to predict the missing two numbers in a consistent way, i. e. in such a way that the mean values of all correlation measurements (actual and hypothetical ones) approach the mean values of the actual correlation measurements alone. In the words of A. Peres [9]: *Unperformed experiments have no results.*

Next we introduce the correspondence to spin systems, see table I. The basic idea is to re-interpret (8) as a statement about the energy of a classical Heisenberg spin system Σ of four spins, which are represented by the unit vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$. Of course, Σ has nothing to do with the quantum two-spin system for which the EPR type of measurements are performed. The different signs in (8) reflect the coupling between the four spins: Σ can be visualized as a square with three anti-ferromagnetic bonds ($J = +1$) and one ferromagnetic bond ($J = -1$), see (39) and figure 3. Each spin configuration $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ realizing the lower bound of (8) is thus a classical ground state for the (dimensionless) Hamiltonian

$$H(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{d} + \vec{c}\cdot\vec{b} - \vec{c}\cdot\vec{d}\,. \tag{10}$$

In this paper we will always denote the ground state energy of a spin system by $-E_0$ in order to avoid inequalities of the form $E_0 \leq \ldots \leq -E_0$.
The ground states do not minimize each term in (10) separately. Indeed, each term in (10) has the form of a spin dimer with a classical ground state energy of -1, irrespective of the sign of the dimer term. Adding these four ground state energies would result in $-E_0 = -4$ (the same as in the case where all four signs in (10) are $+1$). Actually, the ground state energy of (10) is only $-E_0 = -2\sqrt{2}$ since one cannot minimize each term of (10) separately without regard to the other terms.
Spin systems with this property, namely where not each term in the Heisenberg (or Ising) Hamiltonian can be minimized simultaneously by a classical ground state will be called "frustrated" throughout this article. Sometimes this notion is used in the literature with slightly different meanings. For the theory of frustrated spin systems, see, for example, [18], or, for an introduction, [10]. One of the simplest examples is the anti-ferromagnetic (AF) spin triangle where the local states $\uparrow\downarrow$ minimize one term in the Heisenberg Hamiltonian but cannot be extended to a global ground state. The AF triangle has coplanar ground states with angles of 120° between the spin vectors and is said to be "geometrically frustrated". The frustrated square (10), or "Bell square", as we will call it, is an example of non-geometric frustration.

The Bell square is an "AB-system", that is, the spins can be divided into two sub-lattices $\mathcal{A}$ and $\mathcal{B}$ such that only $\mathcal{A}$-spins interact with $\mathcal{B}$-spins. (The nomenclature is again reminiscent of "Alice" and "Bob"). A system with this property, non-positive interaction *w*ithin the sub-lattices and non-negative interaction *b*etween the sub-lattices is usually called a "bi-partite" system and has ground states of the form that, say, all $\mathcal{A}$-spins are down and all $\mathcal{B}$-spins are up. The Bell square is an AB-system, but not bi-partite. This could be taken as a preliminary definition of "non-geometric frustration". However, we will not attempt to define "geometrical frustration" and "non-geometrical frustration" more precisely and will rather use these notions in a somewhat intuitive and vague sense.

Now we can also re-interpret the CHSH inequality, or, rather, its precursor (1), as a statement about the Bell square spin system: It simply says that its energy according to the Ising model is bounded by ± 2. In the Ising model, the individual spin is not represented by a 3-dimensional unit vector, but, so to speak, by an 1-dimensional unit vector $\uparrow$ or $\downarrow$, or, equivalently, by numbers ± 1. The bi-partite systems mentioned above have ground states of the form $\uparrow$, $\downarrow$, i. e. collinear or Ising ground states. But the Bell square Heisenberg spin system has a coplanar ground state with a ground state energy $-E_0 = -2\sqrt{2}$ below its Ising model ground state energy $-E_0^{(I)} = -2$. This is the violation of the CHSH inequality in QT, translated into the language of spin systems.

We will try to exploit the described correspondence in order to construct GBI's violated by the entangled singlet state in QT by considering the corresponding non-geometrically frustrated spin systems with only coplanar or 3-dimensional ground states. To this end we need some more general notation which will be adapted to the described correspondence.

The state of the spin system Σ is described by N unit vectors $\vec{s}_\mu,\ \mu = 1,\ldots,N$. Moreover, the set of spins is divided into two disjoint subsets, $\{1,\ldots,N\} = \mathcal{A} \dot{\cup} \mathcal{B}$, such that the Hamiltonian of Σ can be written in the form

$$H = \sum_{\mu\in\mathcal{A}} \sum_{\nu\in\mathcal{B}} J_{\mu\nu} \vec{s}_\mu \cdot \vec{s}_\nu \ , \tag{11}$$

where the $J_{\mu\nu}$ are real coupling coefficients. The minimum of the Hamiltonian (11) is called the ground state energy $-E_0$.
The corresponding Ising model has states described by numbers $s_\mu = \pm 1,\ \mu = 1,\ldots,N$ and an Ising Hamiltonian

$$H^{(I)} = \sum_{\mu\in\mathcal{A}} \sum_{\nu\in\mathcal{B}} J_{\mu\nu} s_\mu s_\nu \tag{12}$$

with the minimum $-E_0^{(I)}$, the Ising ground state energy. In both cases, the values of the Hamiltonian change their sign under the spin flip transformation $\vec{s}_\mu \mapsto -\vec{s}_\mu$, resp. $s_\mu \mapsto -s_\mu$, $\mu \in \mathcal{A}$.

In the context of the EPR situation, the $s_\mu = \pm 1,\ \mu = 1,\ldots,N$ are the possible outcomes of one experiment and the GBI assumes the form

$$-E_0^{(I)} \le \sum_{\mu\in\mathcal{A}} \sum_{\nu\in\mathcal{B}} J_{\mu\nu} \langle s_\mu s_\nu \rangle \le E_0^{(I)} \ . \tag{13}$$

The N unit vectors $\vec{s}_\mu,\ \mu \in \mathcal{A}$ describe the directions of spin measurements at, say, the left-hand particle (done by Alice), and analogously $\vec{s}_\mu,\ \mu \in \mathcal{B}$ for measurements at the right-hand particle (done by Bob). The quantum theoretical counter-part of (13) is the inequality

$$-E_0 \le -\sum_{\mu\in\mathcal{A}} \sum_{\nu\in\mathcal{B}} J_{\mu\nu} \langle A_\mu B_\nu \rangle = \sum_{\mu\in\mathcal{A}} \sum_{\nu\in\mathcal{B}} J_{\mu\nu} \vec{s}_\mu \cdot \vec{s}_\nu \le E_0 \ , \tag{14}$$

where all correlations are calculated in the singlet state (4). The GBI (13) is hence violated in QT if and only if $E_0^{(I)} < E_0$.

III. CALCULATION OF THE GROUND STATE

For the calculation of the classical ground state of a spin system Σ there exists no straightforward method. However, sometimes the following considerations are useful. For more details see [11]. We write the coupling constants $J_{\mu\nu}$ as the entries of a symmetric $N \times N$-matrix $\mathbb{J}$. Let $j_{\min}$ denote its lowest eigenvalue. Then the Rayleigh-Ritz variation principle yields

$$2H = \sum_{\mu,\nu=1}^{N} J_{\mu\nu} \vec{s}_\mu \cdot \vec{s}_\nu \ge j_{\min} \sum_{\mu=1}^{N} (\vec{s}_\mu)^2 = N j_{\min} \ , \tag{15}$$

whence

$$\frac{1}{2} N j_{\min} \le -E_0 \ . \tag{16}$$

In general, this is only a lower bound and we may have $\frac{1}{2} N j_{\min} < -E_0$. However, if $(s_\mu^{(i)})_{\mu=1,\ldots,N},\ i = 1,2$ are two linearly independent eigenvectors of $\mathbb{J}$ with eigenvalue $j_{\min}$ such that $(s_\mu^{(1)})^2 + (s_\mu^{(2)})^2 = 1$ for all $\mu = 1,\ldots,N$ then the inequality (15) shows that we have found a coplanar ground state

$$\vec{s}_\mu = \begin{pmatrix} s_\mu^{(1)} \\ s_\mu^{(2)} \end{pmatrix}, \ \mu = 1, \ldots, N \tag{17}$$

with ground state energy $-E_0 = \frac{1}{2} N j_{\min}$. Analogously we can argue for three linearly independent eigenvectors which yield a 3-dimensional ground state of H.

To find the Ising ground state $(s_\mu)_{\mu=1,\ldots,N}$, the simplest method would be to check all 2^N Ising spin configurations. Since we can choose, say, $s_1 = 1$ without loss of generality, it would suffice to check 2^{N-1} states. But also this can be a forbidding large number if N is not too small. Assume, for example, that we have $N = 32$ spins as in section V C and that the calculation of the Ising energy of a single state and the comparison with the minimum previously obtained requires approximately 0.01 seconds for a program on a desktop computer. Then the total number of 2^{31} calculations would already last longer than eight months. There exist sophisticated methods to find Ising states which represent a local, rather low energy minimum, see e. g. [12], but for these methods we cannot be sure that we have found the global minimum. In our case of weakly bi-partite systems the following simplification is possible: It suffices to check all Ising states of a subsystem, say, $s_\mu, \ \mu \in \mathcal{A}$. If the $\mathcal{A}$-spins are fixed, the remaining spins $s_\nu, \ \nu \in \mathcal{B}$ are calculated according to

$$s_\nu = -\text{sign}\left(\sum_{\mu \in \mathcal{A}} J_{\mu\nu} s_\mu \right) \ \text{ for all } \nu \in \mathcal{B} \ . \tag{18}$$

The sign according to (18) minimizes the energy of the interaction of the ν-th Ising spin with its neighbors and hence must be assumed for the total Ising ground state. This simplification reduces in our above example 2^{31} calculations to 2^{15} ones and thus the time for the total calculation from months to minutes.

An alternative method to calculate the exact Ising ground state is the so-called"branch and bound" method, see [13]. According to this method the ground state problem is viewed as the problem of finding a minimum (or maximum) of a function defined on the binary tree given by the possible signs of the individual Ising spins, such that good trial state energy or "bound" for the minimum is available. When scanning through the various possibilities of the tree, one can "cut" those branches of the tree which will never reach the bound, even if the most optimistic expectation for the remaining energies is adopted. This results in a considerable reduction of the calculation time for finding the absolute minimum of the energy, see [13].

IV. GENERATING GENERALIZED BELL INEQUALITIES

In this section we describe a recipe how to construct new GBI's and provide a test whether they are violated by the singlet state in QT. We proceed by constructing frustrated spin systems.

- We start by choosing an integer N and a partition of $\{1, \ldots, N\}$ into two disjoint finite sets $\mathcal{A}$ and $\mathcal{B}$, not necessarily with the same number of elements. These sets correspond to the bipartition of the spin system or to possible measurements performed by Alice and Bob.
- Then we choose some real coefficients $J_{\mu\nu}, \ \mu \in \mathcal{A}, \ \nu \in \mathcal{B}$. They can be arbitrary but it is not advisable to choose all coefficients with the same sign since we are seeking for frustrated spin systems.
- Next we find an Ising ground state, i. e. a sequence $s_\mu = \pm 1, \ \mu = 1, \ldots, N$ minimizing the energy $H^{(I)} = \sum_{\mu \in \mathcal{A}} \sum_{\nu \in \mathcal{B}} J_{\mu\nu} \, s_\mu \, s_\nu$. This can be done by using the procedure described in the previous section III. Let $-E_0^{(I)}$ denote the Ising ground state energy. We thus obtain a GBI of the form

$$-E_0^{(I)} \leq \sum_{\mu \in \mathcal{A}} \sum_{\nu \in \mathcal{B}} J_{\mu\nu} \, \langle s_\mu \, s_\nu \rangle \leq E_0^{(I)} \ . \tag{19}$$

- We perform a spin flip transformation (22),(23) such that the Ising ground state becomes $\uparrow\uparrow \ldots \uparrow$. Denote the transformed coefficients again by $J_{\mu\nu}$.
- We define a symmetric matrix $\mathbb{J}$ with $J_{\mu\nu} = J_{\nu\mu}$ as non-diagonal elements. The diagonal elements of $\mathbb{J}$ are chosen in such a way that $\mathbb{J}$ will have constant row sums j and vanishing trace, see [11]. This leaves the Hamiltonian (11) unchanged. Consequently, the Ising ground state $(1, 1, \ldots, 1)$ will be an eigenvector of $\mathbb{J}$ with eigenvalue j.

- We calculate the lowest eigenvalue $j_{\rm min}$ of $\mathbb{J}$. If $j_{\rm min} = j$ the Ising ground state is already the Heisenberg ground state of the spin system and our GBI will not be violated in QT. We have to start the search anew. If, however, $j_{\rm min} < j$ we are done: We have found a GBI which is violated by the singlet state in QT.

The single steps of this recipe are more or less obvious except the last one. We know from (16) that $j_{\rm min} < j$ is a necessary condition for a spin system to have a ground state energy $-E_0 < -E_0^{(I)}$, but is it also sufficient?

In order to prove this we choose the fully polarized Ising ground state to point into the 3-direction of our coordinate frame. Let $(x_\mu)_{\mu=1\ldots,N}$ denote an eigenstate of $\mathbb{J}$ with eigenvalue $j_{\rm min}$. Since, by assumption, $j_{\rm min} < j$ this eigenstate is orthogonal to the Ising ground state, i. e. $\sum_{\mu=1}^N x_\mu = 0$. Next we consider a smooth curve in the state space of our spin system, i. e. a set of unit vector functions $t \mapsto \vec{s}_\mu(t),\ \mu = 1\ldots,N$ satisfying

$$\vec{s}_\mu(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \frac{d}{dt}\vec{s}_\mu(0) = \begin{pmatrix} x_\mu \\ 0 \\ 0 \end{pmatrix},\ \mu = 1\ldots,N\,. \tag{20}$$

The Taylor expansion at $t = 0$ of the Hamiltonian evaluated at the states of this curve, $H(t)$, yields, after a straight-forward calculation, using $\sum_{\mu=1}^N x_\mu = 0$ and $\frac{d^2}{dt^2}\vec{s}_\mu(0)\cdot\vec{s}_\mu(0) + \frac{d}{dt}\vec{s}_\mu(0)\cdot\frac{d}{dt}\vec{s}_\mu(0) = 0$,

$$H(t) = H(0) + \frac{t^2}{2}(j_{\rm min} - j)\sum_{\mu=1}^N x_\mu^2 + \mathcal{O}(t^3)\,. \tag{21}$$

Hence $H(t) < H(0)$ for sufficiently small t and $-E_0^{(I)}$ cannot be the ground state energy. This concludes the proof of the above statement.
Note, that we didn't prove that the energy $\frac{N}{2}j_{\rm min}$ is assumed by some ground state. In general, this will not be the case. The proof only shows that $\frac{N}{2}j_{\rm min} \le -E_0 < -E_0^{(I)} = \frac{N}{2}j$.

V. EXAMPLES

According to section IV arbitrary many GBI's can be constructed. Nevertheless, it will be instructive to show how known examples of GBI's fit into our scheme and to consider further examples for which the ground states can be calculated in closed form.

A. The frustrated $2n$-ring

As a generalization of the frustrated Bell square with $N = 4$ we consider a spin ring with an even number $N = 2n$ of spins, cyclic boundary condition $N + 1 \equiv 1$ and nearest neighbor interaction. Let us for a moment also consider the bi-partite AF system with $J = 1$. It has a ground state $\uparrow\downarrow\uparrow\downarrow\ldots\downarrow$ which is also an Ising ground state. The energy is unchanged by a spin flip transformation

$$\vec{s}_\mu \mapsto \delta_\mu \vec{s}_\mu \tag{22}$$
$$J_{\mu\nu} \mapsto \delta_\mu\delta_\nu J_{\mu\nu}\,, \tag{23}$$

where $\mu,\nu = 1,\ldots,N$ and $\delta_\mu = \pm 1$. Hence the transformed ground state will be the ground state of the transformed Hamiltonian. The spin flip transformation does not alter the number of -1-signs in the ring mod 2: After the transformation there exists an even number of -1-signs, compared to zero -1-signs before the transformation. Obviously, any distribution of an even number of -1-signs in the ring can be obtained by a suitable spin flip transformation starting from the the bi-partite AF system with $J = 1$. Similarly, all distributions of an odd number of -1-signs are equivalent up to spin flip transformations. Hence it will be enough to consider only one of these, say, the ring with one negative coupling constant between N and 1:

$$J_{\mu\nu} = \begin{cases} 1 & \text{if} \quad \nu = \mu + 1 \text{ and } \mu \neq N \\ -1 & \text{if} \quad \mu = N \text{ and } \nu = 1 \\ 0 & \text{else} \end{cases} \tag{24}$$

This system will be called the frustrated $2n$-ring. Its Ising energy satisfies

$$|s_1 s_2 + s_2 s_3 + \dots s_{N-1} s_N - s_N s_1| \leq N - 2 \; , \tag{25}$$

since not all terms in (25) can be positive. The Ising ground state energy $-E_0^{(I)} = -N+2$ is realized by the alternating state $\uparrow\downarrow\uparrow \dots \downarrow$.
The Heisenberg ground state energy $-E_0$ is strictly lower. It is assumed by any spin configuration satisfying

$$\sphericalangle(\vec{s}_\mu, \vec{s}_{\mu+1}) = \pi \frac{N-1}{N}, \; \mu = 1, \dots, N-1 \; . \tag{26}$$

It follows that

$$\sphericalangle(\vec{s}_1, \vec{s}_N) = \pi \frac{(N-1)^2}{N} = \frac{\pi}{N} \quad \mod 2\pi \; , \tag{27}$$

since N is even. Hence the energy of this configuration is

$$H_0 = \sum_{\mu=1}^{N-1} \vec{s}_\mu \cdot \vec{s}_{\mu+1} \; - \vec{s}_N \cdot \vec{s}_1 \tag{28}$$

$$= (N-1) \cos\left(\pi \frac{N-1}{N}\right) - \cos\frac{\pi}{N} \tag{29}$$

$$= -N \cos\frac{\pi}{N} = -N + \frac{\pi^2}{2N} - \frac{\pi^4}{24 N^3} \pm \dots \tag{30}$$

$$< 2 - N \text{ for } N \geq 4 \; . \tag{31}$$

According to the remarks in section III it is possible to prove that (26) is a ground state by calculating the lowest eigenvalue of the matrix $\mathbb{J}$ corresponding to the coupling constants (24). To this end we define the orthogonal $N \times N$-matrix $\mathbb{U}$ by

$$U_{\mu\nu} = \begin{cases} 1 & \text{if} \quad \nu = \mu + 1 \text{ and } \mu \neq N \\ -1 & \text{if} \quad \mu = N \text{ and } \nu = 1 \\ 0 & \text{else} \end{cases} \tag{32}$$

It follows that $\mathbb{J} = \mathbb{U} + \mathbb{U}^*$ and $[\mathbb{J}, \mathbb{U}] = 0$, hence the eigenvalues of $\mathbb{J}$ are obtained as twice the real part of the eigenvalues of $\mathbb{U}$. The latter are of the form

$$u_k = \exp\left(\frac{i\,\pi\,k}{N}\right), \; k = 1, 3, \dots, 2N-1 \; , \tag{33}$$

since $\mathbb{U}^N = -\mathbb{I}_N$. Hence the eigenvalues of $\mathbb{J}$ are

$$j_k = 2\cos\left(\frac{\pi\,k}{N}\right), \; k = 1, 3, \dots, 2N-1 \; , \tag{34}$$

and the lowest eigenvalue $j_{\min}$ is obtained for $k = N-1$. By (15) this yields the bound $H \geq \frac{N}{2} j_{\min} = -N \cos\frac{\pi}{N}$, which confirms that (30) is the minimal energy of the frustrated $2n$-ring.
The GBI (25) has been found by P. M. Pearle [5] and its violation (31) by quantum correlations has been discussed by S. L. Braunstein and C. M. Caves [6] without proving the maximal violation for the configuration (26).

B. The frustrated hexagon

Frustrated even rings are not the only AB-systems. We consider as another example the frustrated hexagon with additional interactions between even and odd spin sites, see figure 1. Its coupling constants are chosen as follows:

$$J_{12} = J_{14} = J_{25} = -1 \tag{35}$$

$$J_{23} = J_{34} = J_{45} = J_{56} = J_{16} = J_{36} = 1 \; . \tag{36}$$

The corresponding GBI reads

$$-5 \leq \langle s_2 s_3 \rangle + \langle s_3 s_4 \rangle + \langle s_4 s_5 \rangle + \langle s_5 s_6 \rangle + \langle s_6 s_1 \rangle + \langle s_3 s_6 \rangle - \langle s_1 s_2 \rangle - \langle s_1 s_4 \rangle - \langle s_2 s_5 \rangle \leq 5 \; . \tag{37}$$

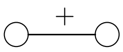
corresponding to the Ising ground state $\uparrow\uparrow\uparrow\downarrow\uparrow\downarrow$. The spin configuration with

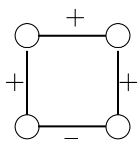
$$\vec{s}_1 = \vec{s}_2, \; \sphericalangle(\vec{s}_6, \vec{s}_4) = \sphericalangle(\vec{s}_4, \vec{s}_2) = \sphericalangle(\vec{s}_2, \vec{s}_5) = \sphericalangle(\vec{s}_5, \vec{s}_3) = 60^\circ \; , \tag{38}$$

see figure 2, realizes the Heisenberg ground state with energy $-E_0 = -6$ corresponding to a violation of (37) in QT. Since this result can be confirmed by using the same methods as in section V A, we leave the details to the reader.

C. Frustrated hypercubes

Another way to generalize the Bell square is to consider "frustrated hypercubes" H_n with $N = 2^n$ vertices and $n = 1, 2, 3, \ldots$. The N spins are located at the vertices and are interacting along the $n 2^{n-1}$ edges of the hypercube. Sometimes, the vertices $v_\nu, \; \nu = 0, \ldots, N-1$ are most conveniently labelled by binary strings $(\delta_1, \ldots, \delta_n)$ of length n, such that the $\delta_i \in \{0, 1\}$ are the binary digits of ν. Two strings v, w which differ exactly at one position form an edge $e = \{v, w\}$. The set of vertices can be divided into the set $\mathcal{A}$ of binary strings with an even number of ones and the set $\mathcal{B}$ of binary strings with an odd number of ones. Thus every edge connects an $\mathcal{A}$-vertex with a $\mathcal{B}$-vertex and the corresponding spin system is an AB-system. The sign attached to an edge $e = \{v, w\}$ can be defined as $(-1)^\ell$ where ℓ is the number of ones at the left hand side of the position where v and w differ.

Alternatively, the signs can be defined recursively according to the following procedure.
If the signs of H_n are already defined, denote by H_n^* the same hypercube with inverted signs. Define H_{n+1} as the union of H_n and H_n^* where N new edges between the corresponding vertices of H_n and H_n^* are added and $+1$-signs are attached to them.
For example, if we start with $H_1 =$  and apply this procedure we obtain

$H_2 =$  (39)

which is just the Bell square considered before. See figure 3 for the frustrated hypercubes H_3 and H_4 obtained by this procedure. Of course, one can apply symmetry transformations of H_n (suitable rotations and reflections) and spin flip transformations to obtain modified frustrated hypercubes. But these will have analogous properties as the H_n and need not be considered further.
The recursive procedure $H_n \rightarrow H_{n+1}$ can also be applied to the $\mathbb{J}$-matrices. Indeed, $\mathbb{J}_n$ can be recursively defined by

$$\mathbb{J}_0 = (0) \tag{40}$$

$$\mathbb{J}_{n+1} = \begin{pmatrix} \mathbb{J}_n & \mathbb{I}_N \\ \mathbb{I}_N & -\mathbb{J}_n \end{pmatrix} \; , \tag{41}$$

where $\mathbb{I}_N$ denotes the $N \times N$ identity matrix. By induction we conclude $\mathbb{J}_n^2 = n \mathbb{I}_N$ and hence the eigenvalues of $\mathbb{J}_n$ are $\pm\sqrt{n}$ with degeneracies $\frac{N}{2}$. If these eigenvalues can be realized by spin vector configurations the corresponding ground state energy $-E_0^{(n)} = -\frac{N}{2}\sqrt{n}$ must be lower than the Ising ground state energy $-E_0^{(n,I)}$ which is always an integer. Possible exceptions are $n = 4, 9, 16, \ldots$ where $E_0^{(n)}$ is an integer too.

We cannot present a general theorem which warrants that the ground state energy $-E_0^{(n)} = -\frac{N}{2}\sqrt{n}$ will be assumed by a coplanar state for all $n \geq 2$. Rather we have studied the cases $n = 3, 4, 5$ in some detail with the following results.
For $n = 3$ and $n = 5$ we have explicitly constructed coplanar spin configurations realizing the ground state energy $-E_0^{(n)} = -\frac{N}{2}\sqrt{n}$ which is below the Ising ground state energy. In these cases we therefore obtain GBI's which are violated in QT. For $n = 4$ the ground state energy $-E_0^{(4)} = -16$ is realized by an Ising state. Hence we can also in this case write down a GBI, but it is also satisfied in QT and hence of no use in the context of the EPR discussion.

This GBI is also not violated for other entangles states, see section VI.

The frustrated hypercubes H_n have an obvious reflection symmetry σ defined by $\nu \stackrel{\sigma}{\longleftrightarrow} \nu+1, \nu$ even. We thus have $J_{\sigma(\mu)\sigma(\nu)} = J_{\mu\nu}$. This can by proven by induction over n using the recursive construction procedure $H_n \to H_{n+1}$. Hence the eigenvectors of $\mathbb{J}$ can be chosen either to be invariant under σ or to change their sign. Recall that the x- and the y-components of the coplanar spin configuration realizing the ground state energy $-E_0^{(n)} = -\frac{N}{2}\sqrt{n}$ are eigenvectors of $\mathbb{J}$. It turns out that we can always find ground state configurations

$$\vec{s}_\mu = \begin{pmatrix} x_\mu \\ y_\mu \end{pmatrix} \text{ satisfying } x_{\sigma(\mu)} = x_\mu \text{ and } y_{\sigma(\mu)} = -y_\mu \text{ for } \mu = 1, \ldots, N \; . \tag{42}$$

Hence it will suffice to write down the ground state configurations $\vec{s}_\mu$ only for even μ.

All components of $\vec{s}_\mu$ can be written as radicals with a similar structure consisting of nested square roots and small integers. In order to simplify table II containing the ground state configurations we hence introduce the following abbreviations:

$$n = 2 \;:\; V_2(\delta_1, \delta_2, \delta_3) \equiv \frac{1}{2} \begin{pmatrix} \delta_1 \sqrt{2 + \delta_3 \sqrt{2}} \\ \delta_2 \sqrt{2 - \delta_3 \sqrt{2}} \end{pmatrix} \tag{43}$$

$$n = 3 \;:\; V_3(\delta_1, \delta_2, \delta_3, \delta_4) \equiv \frac{1}{2} \begin{pmatrix} \delta_1 \sqrt{2 + \delta_3 \sqrt{\frac{2}{3}} + \delta_4 \frac{2}{\sqrt{3}}} \\ \delta_2 \sqrt{2 - \delta_3 \sqrt{\frac{2}{3}} - \delta_4 \frac{2}{\sqrt{3}}} \end{pmatrix} \tag{44}$$

$$n = 4 \;:\; V_4(\delta_1, \delta_2) \equiv \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \tag{45}$$

$$n = 5 \;:\; V_5(\delta_1, \delta_2, \delta_3) \equiv \begin{pmatrix} \delta_1 \sqrt{\frac{1}{2} + \frac{1}{\delta_3 \sqrt{5}}} \\ \delta_2 \sqrt{\frac{1}{2} - \frac{1}{\delta_3 \sqrt{5}}} \end{pmatrix} \; . \tag{46}$$

The coplanar ground states for some frustrated hypercubes H_n are visualized in the figures 4 ($n = 2$), 5 ($n = 3$) and 6 ($n = 5$).

VI. GENERAL ENTANGLED STATES

In the correspondence between GBI's and frustrated AB-systems outlined in section II we have always chosen the singlet state $\phi = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ as the state relative to which the correlations $\langle AB \rangle$ etc. have been calculated. The rotational invariance of ϕ then directly corresponds to the rotational invariance of the Heisenberg Hamiltonian (11). This picture changes if other entangled states $\psi \in \mathcal{H} \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$ are taken into account.

Such states can be written as a Schmidt bi-orthogonal sum

$$\psi = \sum_{i=1}^{2} c_i \mathbf{u}_i \otimes \mathbf{v}_i \; , \tag{47}$$

where $\{\mathbf{u}_i\}_{i=1,2}$ and $\{\mathbf{v}_i\}_{i=1,2}$ are orthonormal bases in $\mathbb{C}^2$, and $|c_1|^2 + |c_2|^2 = 1$, see [8], section 5. 3. ψ is an entangled state if and only if $c_1, c_2 \neq 0$. Upon choosing appropriate, possibly different, coordinate frames for Alice and Bob, (47) assumes the form

$$\psi = \cos\alpha |\uparrow\downarrow\rangle - \sin\alpha |\downarrow\uparrow\rangle \; , \tag{48}$$

where $0 < \alpha < \frac{\pi}{2}$. $\alpha = \frac{\pi}{4}$ corresponds to the singlet state ϕ. To put this in another way, ψ can be written in the form (48) by means of a unitary transformation in $\mathcal{H}$.

After a straightforward calculation we obtain for the correlation of observables of the form (5)

$$\langle AB \rangle \equiv \langle \psi | A \otimes B | \psi \rangle \tag{49}$$

$$= -a_3 b_3 - \sin(2\alpha)(a_1 b_1 + a_2 b_2) \tag{50}$$

$$\equiv -a_3 b_3 - \gamma(a_1 b_1 + a_2 b_2) \; . \tag{51}$$

The corresponding classical spin Hamiltonian for a general frustrated AB-system reads

$$H(\gamma) = \sum_{\mu\in\mathcal{A}} \sum_{\nu\in\mathcal{B}} J_{\mu\nu} \left(z_\mu z_\nu + \gamma(x_\mu x_\nu + y_\mu y_\nu) \right) \ , \tag{52}$$

where we have written

$$\vec{s}_\mu \equiv \begin{pmatrix} x_\mu \\ y_\mu \\ z_\mu \end{pmatrix} \ \text{for } \mu = 1, \ldots, N \ . \tag{53}$$

Such anisotropic spin Hamiltonians are well-known under the name "XXZ-model", see, for example [15]. For $\gamma \to \infty$ the XXZ-model essentially approaches the Ising model, which means that, in the context of GBI's, factorable states will not violate the GBI.
It has been proven [14],[8] that the CHSH inequality is also violated for an arbitrary entangled state ψ, even for a general Hilbert space $\mathcal{H}$. This means, in the language of spin systems, that the XXZ Bell square has a coplanar ground state for all γ with $0 < \gamma \le 1$. Unfortunately, we have not found a proof of the analogous statement for the class of GBI's considered in this article. We only remark that, obviously, the GBI is violated for all γ with $E_0^{(I)}/E_0 < \gamma < 1$, if $E_0^{(I)} < E_0$ and the GBI is violated for $\gamma = 1$ with a coplanar ground state. Indeed, if we evaluate the Hamiltonian $H(\gamma)$ at the coplanar ground state $(\vec{s}_\mu)_{\mu=1,\ldots,N}$ we obtain

$$H(\gamma) \ = \ \sum_{\mu\in\mathcal{A}} \sum_{\nu\in\mathcal{B}} J_{\mu\nu} (\underbrace{z_\mu z_\nu}_{=0} + \gamma(x_\mu x_\nu + y_\mu y_\nu)) \tag{54}$$

$$= \ \gamma H(1) = -\gamma E_0 < -E_0^{(I)} \ . \tag{55}$$

We now consider the case that the Ising ground state is an eigenstate of $\mathbb{J}$ corresponding to the lowest eigenvalue $j_{\min}$, and hence the GBI is *n*ot violated for the singlet state ϕ. This happens in the example of the frustrated hypercube H_4, see table II. Then the GBI will also not be violated for other entangled states ψ, i. e. , the Ising ground state will remain the ground state for all $H(\gamma)$. To show this we consider

$$2H(\gamma) \ = \ \sum_{\mu,\nu=1}^{N} J_{\mu\nu}(z_\mu z_\nu + \gamma(x_\mu x_\nu + y_\mu y_\nu)) \tag{56}$$

$$\ge \ j_{\min}(\underbrace{\sum_{\mu=1}^{N} z_\mu^2}_{\zeta} + \gamma \underbrace{\sum_{\mu=1}^{N} (x_\mu^2 + y_\mu^2)}_{\xi}) \tag{57}$$

$$\equiv \ j_{\min}(\zeta + \gamma\xi) \ge N j_{\min} \ , \tag{58}$$

since $j_{\min} < 0$ and $\zeta + \gamma\xi$ assumes its maximum under the constraint $\zeta + \xi = N$ for $\zeta = N$. The minimum (58) is assumed by inserting the Ising ground state for the z_μ and setting $x_\mu = y_\mu = 0$.

VII. CONCLUSION

What are the benefits of the proposed correspondence between the considered class of GBI's and frustrated AB-systems?

For readers which are mainly interested in the EPR discussion and Bell inequalities the most interesting result might be the recipe to obtain an arbitrary number of GBI's, see section IV. This construction procedure should give us some additional insight into the structure of GBI's. Moreover, we hope that the proposed correspondence would lead the reader to think of GBI's in a more geometric or graphical way: The possible measurements can be visualized as vertices of a spin system and the signed correlations occurring in the GBI as the edges or bonds of this system. As we have shown, some methods from the theory of spin systems can be employed to tackle questions in the realm of the foundations of QT.

Also for readers which are mainly interested in frustrated spin systems, the unexpected connection to GBI's might be valuable in its own right. For problems in the theory of spin systems a transfer of methods from another field of research will be of some interest. Moreover, those readers will probably find the class of frustrated hypercubes, section V C, to be a useful set of toy examples worth while to be further studied. For example, in the quantum XY-model of frustrated hypercubes there exist so-called localized multi-magnon states which lead to prominent properties as huge magnetization jumps and large residual entropy at $T = 0$ resulting in a marked magneto-caloric effect, see, for example [16], [17], and [19].
Summarizing, the correspondence between GBI's and frustrated spin systems fosters the insight into both branches of physics and leads to some new phenomena in non-geometric frustration.

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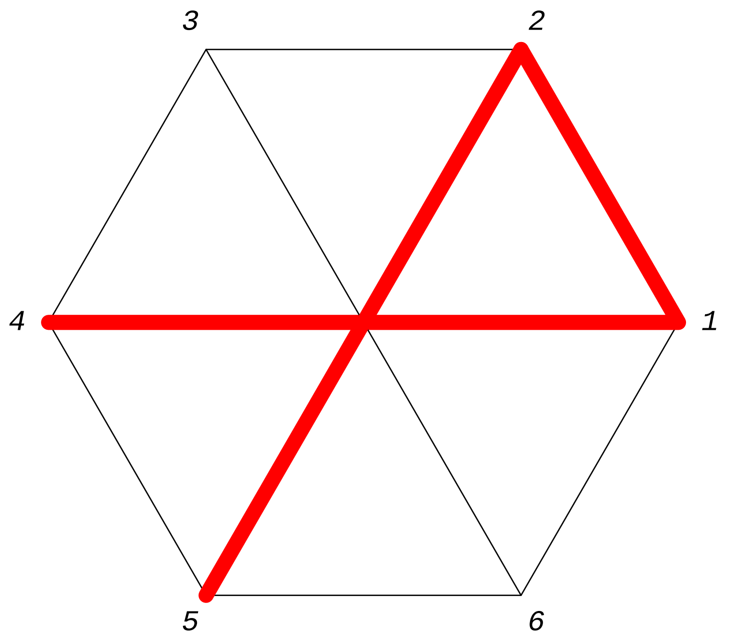

FIG. 1: A frustrated hexagon. The thick red lines indicate the edges with a coupling constant of −1, the thin black lines correspond to the coupling +1.

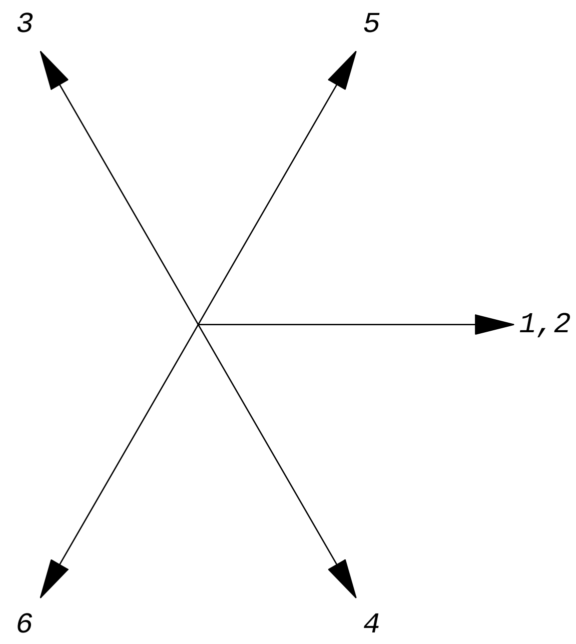

FIG. 2: A coplanar ground state of the frustrated hexagon. The correlations of measurements of the spin components according to these vectors in the singlet state violate the inequality (37).

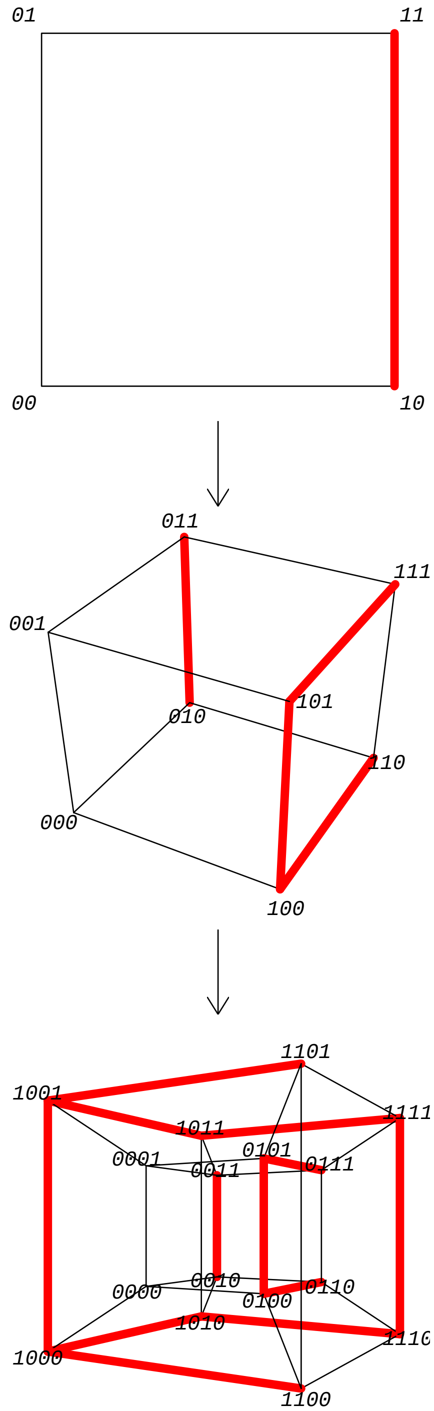

FIG. 3: A sequence of frustrated hypercubes $H_n,\ n = 2, 3, 4$ generated by a recursive procedure, see section V C. The negative bonds are indicated by thick red lines.

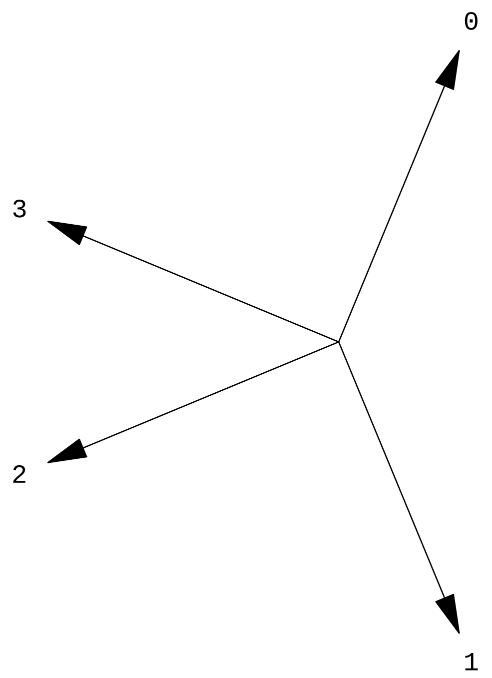

FIG. 4: A coplanar ground state of the frustrated Bell square H_2 according to table II. Note the reflection symmetry of the vectors with label ν and $\nu + 1$, ν even. The correlations of measurements of the spin components according to these vectors in the singlet state violate the CHSH inequality.

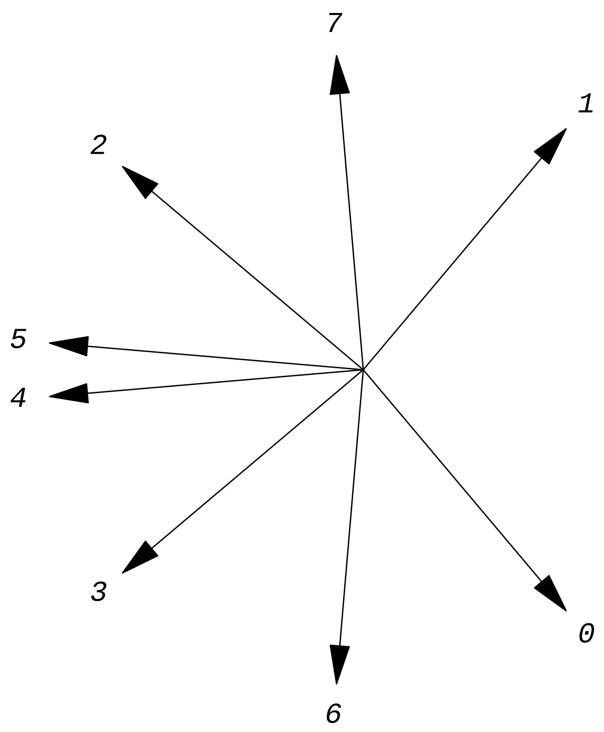

FIG. 5: A coplanar ground state of the frustrated cube H_3 according to table II. Note the reflection symmetry of the vectors with label ν and $\nu + 1$, ν even. The correlations of measurements of the spin components according to these vectors in the singlet state violate the corresponding generalized Bell inequality.

TABLE I: Correspondence between GBI's and classical spin systems.

EPR experiment	Classical spin system
N possible measurements	N spins
2-particle states	bi-linear Hamiltonian
isotropic singlet state	isotropic Heisenberg Hamiltonian $H(\mathbf{s})$
general entangled state	XXZ-Hamiltonian, section VI
2 experimenters (Alice and Bob)	AB-systems
GBI	inequality for Ising states s
$\lvert \sum_{\mu\in\mathcal{A}} \sum_{\nu\in\mathcal{B}} J_{\mu\nu} \langle s_\mu s_\nu \rangle \rvert \le E_0^{(I)}$	$\lvert H^{(I)}(s) \rvert \le E_0^{(I)}$
Violation of the GBI for quantum correlations	non-collinear ground state state $\mathbf{s}$
$\sum_{\mu\in\mathcal{A}} \sum_{\nu\in\mathcal{B}} J_{\mu\nu} \vec{s}_\mu \cdot \vec{s}_\nu = -E_0$	$H(\mathbf{s}) = -E_0$
such that $E_0^{(I)} < E_0$	such that $E_0^{(I)} < E_0$

TABLE II: Results for the ground states of some frustrated hypercubes $H_n,\ n = 2, 3, 4, 5$. The spin vectors $\vec{s}_\mu$ of the ground state are only given for $\mu = 0, 2, \ldots, N-2$; the other spin vectors result from the symmetry (42). Moreover, we utilize the abbreviations (43), (44), (45), and (46). The Ising ground states are degenerate; the table contains only one example of an Ising ground state for each $n = 2, 3, 4, 5$. We observe that for $n = 2, 3, 5$ the Heisenberg ground state energy $-E_0^{(n)}$ is below the energy $-E_0^{(n,I)}$ of the Ising ground state. This corresponds to a violation of the generalized Bell inequalities in quantum theory. For $n = 4$ both ground state energies coincide, hence the corresponding generalized Bell inequality is *not* violated in quantum theory.

n	$N = 2^n$	Heisenberg ground state	$-E_0^{(n)}$	Ising ground state	$-E_0^{(n,I)}$
2	4	$\vec{s}_\mu = V_2(\delta_1, \delta_2, \delta_3)$, where $(\delta_1, \delta_2, \delta_3) =$ $(++-), (--+)$	$-2\sqrt{2} \approx -2.82843$	$\uparrow\downarrow\downarrow\uparrow$	-2
3	8	$\vec{s}_\mu = V_3(\delta_1, \delta_2, \delta_3, \delta_4)$, where $(\delta_1, \delta_2, \delta_3, \delta_4) =$ $(+-+-), (-+-+), (--++), (----)$	$-4\sqrt{3} \approx -6.9282$	$\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\uparrow$	-6
4	16	$\vec{s}_\mu = V_4(\delta_1, \delta_2)$, where $(\delta_1, \delta_2) =$ $(-10), (+10), (+10), (0-1),$ $(+10), (0+1), (0-1), (0+1)$	$-8\sqrt{4} = -16$	$\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow$	-16
5	32	$\vec{s}_\mu = V_5(\delta_1, \delta_2, \delta_3)$, where $(\delta_1, \delta_2, \delta_3) =$ (+, −, -1), (−, −, 2), (+, +, 2), (+, +, -2), (+, +, -1), (−, +, -1), (−, +, -1), (−, −, 2), (−, +, 1), (+, +, -2), (−, −, -2), (+, +, 2), (−, −, 1), (+, −, 1), (+, −, 1), (+, +, -2)	$-16\sqrt{5} \approx -35.7771$	$\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow$	-32

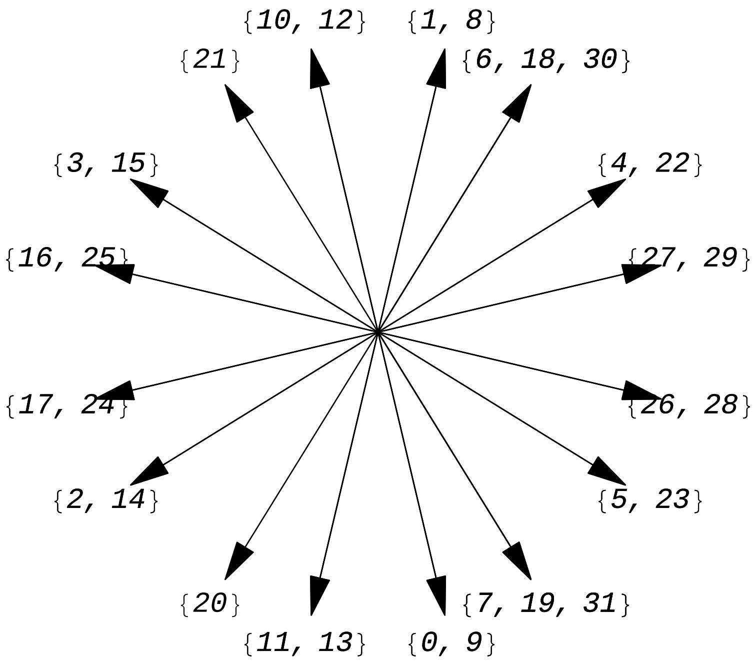

FIG. 6: A coplanar ground state of the frustrated hypercube H_5 according to table II. Note the reflection symmetry of the vectors with label ν and $\nu + 1$, ν even. The correlations of measurements of the spin components according to these vectors in the singlet state violate the corresponding generalized Bell inequality.