

A solvable model of a one-dimensional quantum gas with pair interaction

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Abstract

We propose a solvable model of a one-dimensional harmonic oscillator quantum gas of two sorts of particles, fermions or bosons, which allows to describe the formation of pairs due to a suitable pair interaction. These pairs we call “pseudo-bosons” since the system can be approximated by an ideal bose gas for low temperatures. We illustrate this fact by considering the specific heat and the entropy function for $N = 8$ pairs. The model can also be evaluated in the thermodynamic limit if the harmonic oscillator potential is suitable scaled.

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1 Introduction

It seems to be generally taken for granted that a compound of an even number of fermions behaves like a boson since any state of $2nN$ fermions has the same permutational symmetry as a state of N bosons. Accordingly, the recent experiments with trapped ultracold gases [1] [2] [3] [4] are viewed as an achievement of Bose-Einstein condensation [5]. However, a moment’s thinking reveals that this cannot be true in an exact sense but at most approximately. Consider an ideal gas of N “bosons”, each “boson” consisting of $2n$ fermions bound together by some attractive force. In the high temperature limit the bounds between the fermions will be broken and the system behaves like an ideal fermi gas at high temperatures. Its energy will thus be $E \sim 2nN f_F k_B T$, where f_F is number of degrees of freedom for a single fermion, e. g. $f_F = 6$

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for fermions sitting in a 3-dimensional harmonic oscillator potential which we assume now for sake of simplicity. This would be consistent with the corresponding formula for a bose gas of N particles if the number of degrees of freedom for a single boson is assumed to be $f_B = 2nf_F$. For intermediate temperatures T which are large with respect to the spacing of 1-particle energies, say $k_B T \gg \hbar\omega$, but small with respect to the binding energy of the fermionic compounds, we would expect a “preliminary” classical limit of the energy, say $E \sim 6Nk_B T$, different from $E \sim 12nNk_B T$. Hence the energy function $E(T)$ of the considered gas cannot coincide with that of an ideal bose gas at medium and high temperature.

More interesting with regard to Bose-Einstein condensation is the low temperature limit. N non-interacting bosons would occupy the same 1-particle ground state for $T \rightarrow 0$, which is impossible for $2nN$ fermions according to the Pauli principle. Hence we expect a non-bose behaviour for “bosons” also for low temperatures. In view of these problems we have scruples about identifying compounds of $2n$ fermions with bosons and will rather call them “pseudo-bosons”.

To be sure, there will be a close resemblance between bosons and pseudo-bosons and the thermodynamic functions of ideal gases of each sort should be approximately the same for some range of temperature, not too low and not too high. To make this statement more precise is a challenge to the foundations of quantum statistical mechanics, in particular, to demonstrate the possibility of pseudo-Bose-Einstein condensation. We do not know of any attempt to tackle this problem, with the exception of an estimate of the critical density of a pseudo-Bose-Einstein condensate where fermionic effects would occur [6]. As a first step to treating the problem of pseudo-bosons we seek for simple models describing the formation of pairs of fermions, being the simplest case of pseudo-bosons, which are solvable in the sense that the partition function of the system can be explicitly calculated. For this purpose the model need not be a realistic description of, say, rubidium atoms in a trap. It rather serves as an example to study the problem of pseudo-bosons in principle.

In section 6 (summary and outlook) we will come back to the problem of pseudo-bosons. Of course, the model we are going to present can be considered in its own sake without reference to the specific problem and could be compared with other 1-dimensional solvable models of quantum systems, see [7]. It turns out that the case of pairs of interacting bosons is completely analogous and can be treated simultaneously, although the discussion of the results will be confined to the fermi case. The model will be defined in section 2 and consists of two sorts of N fermions (resp. bosons) with equally spaced non-degenerate 1-particle energy levels. In this aspect it remotely resembles the Luttinger model [8],[9], which is, however, unitarily and hence thermodynamically equivalent to a system of non-interacting particles. In our model the interaction is introduced in such a way that pairs of fermions of different sort with the same quantum number gain a binding energy V . Hence the diagonalization of the total Hamiltonian is trivial. The calculation of the partition

function in section 3, however, requires some combinatorics. For $N \leq 8$ we calculate the specific heat, using computer-algebraic means, and discuss its properties (section 4). For larger N one has to resort to the thermodynamic limit which is derived in section 5. The convergence to this limit is shown to be rather rapid for $T > 0$.

2 Definition of the model

We will consider two sorts of particles with Hilbert spaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ for 1-particle states. Let $(\phi_n^{(1)})_{n \in \mathbb{N}}$ and $(\phi_n^{(2)})_{n \in \mathbb{N}}$ be orthonormal bases in $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$. Define the 1-particle Hamiltonians by

$$H^{(i)}\phi_n^{(i)} = n\hbar\omega\phi_n^{(i)}, \quad i = 1, 2, \quad (1)$$

which holds for example if all particles are assumed to sit in a 1-dimensional harmonic oscillator (HO) potential and the zero point energy is neglected. Usually we will set $\hbar\omega = 1$ in what follows, but later we consider a scaling law $\omega = \omega(N)$ in the context of the thermodynamic limit $N \rightarrow \infty$.

The interaction between particles of different sorts will have the simple form

$$V^{(12)}\phi_n^{(1)} \otimes \phi_m^{(2)} = -V\phi_n^{(1)} \otimes \phi_m^{(2)}\delta_{nm}, \quad (2)$$

where $V > 0$ is a real parameter. Thus the systems gains the binding energy V if two particles of different sorts have the same quantum numbers and form a compound, say, a ‘‘molecule’’. The molecules can be viewed as pseudo-particles in a HO potential with double spacing $2\hbar\omega$ of the energy values.

We consider N fermions (resp. bosons) of each sort, but particles of different sorts are still assumed to be distinguishable. The total Hilbert space will thus be

$$\mathcal{H}^\pm = \left(\bigotimes_{\nu=1}^N \mathcal{H}^{(1)} \right)^\pm \otimes \left(\bigotimes_{\nu=1}^N \mathcal{H}^{(2)} \right)^\pm, \quad (3)$$

where the $^\pm$ sign stands for the completely symmetric (+), resp. antisymmetric (−) subspace. The model is defined by the Hamiltonian

$$H = \sum_{\nu=1}^N H_\nu^{(1)} + \sum_{\mu=1}^N H_\mu^{(2)} + \sum_{\mu,\nu=1}^N V_{\mu\nu}^{(12)}, \quad (4)$$

where the subscripts μ, ν indicate the factor of the tensor product where the 1- and 2-particle Hamiltonians of the form (1) and (2) act upon. Clearly, (4) is invariant under permutations $\pi \in \mathcal{S}_N \times \mathcal{S}_N$ and will be understood to be restricted to the subspace \mathcal{H}^\pm .

In the fermionic case the eigenstates of H can be chosen as the products of Slater determinants

$$\Phi_{\mathbf{nm}} = \bigwedge_{i=1}^N \phi_{n_i}^{(1)} \otimes \bigwedge_{j=1}^N \phi_{m_j}^{(2)}, \quad (5)$$

where all quantum numbers of the same sort are strictly ordered, $n_1 < n_2 < \dots < n_N$ and $m_1 < m_2 < \dots < m_N$. We will write the latter condition in the compact form $\mathbf{nm} \in I^-$. Analogously one obtains the eigenfunctions in the bosonic case, but here we have $n_1 \leq n_2 \leq \dots \leq n_N$ and $m_1 \leq m_2 \leq \dots \leq m_N$ or, equivalently, $\mathbf{nm} \in I^+$. The corresponding energy eigenvalues are

$$E_{\mathbf{nm}} = \sum_{i=1}^N n_i + \sum_{j=1}^N m_j - V \sum_{i,j=1}^N \delta_{n_i m_j}. \quad (6)$$

3 The partition function

The above defined model is admittedly a rather crude description of a real quantum gas, but it allows an explicit calculation of its partition function without any further approximation. As usual, the partition function is defined by

$$\mathcal{Z}_N^\pm(\beta) = \sum_{\mathbf{nm} \in I^\pm} \exp(-\beta E_{\mathbf{nm}}). \quad (7)$$

In view of (6) we will split the series (7) into a finite number of subseries according to the number of coincidences between the two sequences of quantum numbers \mathbf{n} and \mathbf{m} . More precisely, we will split the total index set I^\pm or, equivalently, the set of eigenstates of H into disjoint subsets according to their “order type”. The order type of \mathbf{nm} comprises the linear ordering of the union of both sequences \mathbf{n} and \mathbf{m} , including the coincidences of quantum numbers. For $N = 2$ and the fermionic case we have 10 different order types, for example $(n_1 < n_2 < m_1 < m_2)$ or $(n_1 = m_1 < m_2 < n_2)$.

In general, every order type can be uniquely characterized by a string s composed of symbols $s_i \in \{0, 1, 2\}$. “ $s_i = 1$ ” means “a quantum number of sort 1”, analogously for “ $s_i = 2$ ”, but “ $s_i = 0$ ” means that “two quantum numbers of different sort coincide”. In the above example, $(n_1 < n_2 < m_1 < m_2) \leftrightarrow (1122)$ and $(n_1 = m_1 < m_2 < n_2) \leftrightarrow (021)$.

Let $N_i(s)$ denote the number of occurrences of the symbol “ i ” in the string s and $|s|$ the length of s . We define \mathcal{A}_N as the set of all finite strings s over the alphabet $\{0, 1, 2\}$ such that $N_1(s) = N_2(s) = N - N_0(s)$. Then the strings s in \mathcal{A}_N are in 1 : 1 correspondence with the order types, denoted by $T^\pm(s)$, of the subsequences of (7). Hence we may write

$$\mathcal{Z}_N^\pm(\beta) = \sum_{s \in \mathcal{A}_N} \sum_{\mathbf{mn} \in T^\pm(s)} \exp(-\beta E_{\mathbf{mn}}) \quad (8)$$

$$= \sum_{N_0=0}^N e^{\beta N_0 V} \sum_{s \in \mathcal{A}_{N, N_0}} \sum_{\mathbf{mn} \in T^\pm(s)} \exp\left(-\beta \left(\sum_i n_i + \sum_j m_j\right)\right). \quad (9)$$

In the last equation we have used the definition

$$\mathcal{A}_{N, N_0} = \{s \in \mathcal{A}_N | N_0(s) = N_0\} \quad (10)$$

and that all strings in \mathcal{A}_{N, N_0} lead to the same interaction energy $-N_0 V$.

Note that the series $\sum_{\mathbf{mn} \in T^\pm(s)}$ is only a $2N - N_0$ -fold series because of the N_0 coincidences between the quantum numbers. In the exponent, however, each term in $\sum_i n_i + \sum_j m_j$ has to be counted once, irrespective of coincidences. Hence we may introduce $2N - N_0$ new summation indices a_i if we account for the double occurrences in the exponent by factors g_i . The condition $a_1 < a_2 < \dots < a_{|s|}$ will be written in the compact form $\mathbf{a} \in J^-$, analogously $a_1 \leq a_2 \leq \dots \leq a_{|s|} \Leftrightarrow \mathbf{a} \in J^+$.

This yields

$$\sum_{\mathbf{mn} \in T^\pm(s)} \exp\left(-\beta \left(\sum_i n_i + \sum_j m_j\right)\right) = \sum_{\mathbf{a} \in J^\pm} \exp\left(-\beta \sum_i g_i(s) a_i\right), \quad (11)$$

where we have used $|s| = 2N - N_0$ and the definition

$$g_i(s) = \begin{cases} 1 & s_i = 1 \text{ or } 2 \\ 2 & s_i = 0 \end{cases}. \quad (12)$$

The remaining evaluation of (11) is straight forward. We introduce as new summation variables the differences

$$d_1 = a_1 \quad (13)$$

$$d_n = a_n - a_{n-1}, \quad n > 1 \quad (14)$$

and use the partial sums

$$G_n(s) = \sum_{i=n}^{|s|} g_i(s) \quad (15)$$

since

$$\sum_n G_n d_n = \sum_n G_n (a_n - a_{n-1}) = \sum_n (G_n - G_{n+1}) a_n = \sum_n g_n a_n. \quad (16)$$

Further let

$$\varepsilon^\pm = \begin{cases} 0 & \text{for bosons(+)} \\ 1 & \text{for fermions(-)} \end{cases} \quad (17)$$

Then we obtain

$$\begin{aligned} \sum_{\mathbf{a} \in J^\pm} \exp\left(-\beta \left(\sum_i g_i(s) a_i\right)\right) &= \sum_{d_1=0}^{\infty} \sum_{d_2=\varepsilon^\pm}^{\infty} \dots \sum_{d_{|s|}=\varepsilon^\pm}^{\infty} \exp\left(-\beta \left(\sum_i G_i(s) d_i\right)\right) \\ &= \sum_{d_1=0}^{\infty} e^{-\beta G_1 d_1} \sum_{d_2=\varepsilon^\pm}^{\infty} e^{-\beta G_2 d_2} \dots \sum_{d_{|s|}=\varepsilon^\pm}^{\infty} e^{-\beta G_{|s|} d_{|s|}} \\ &= \frac{1}{1 - e^{-\beta G_1}} \prod_{n=2}^{|s|} \frac{e^{-\varepsilon^\pm \beta G_n}}{1 - e^{-\beta G_n}}. \end{aligned} \quad (18)$$

The result can be further simplified by utilizing the fact that (19) depends only on the G_n , i. e. on the positions of the symbol “0” in the string s . There are exactly $\binom{2(N-N_0)}{N-N_0}$ strings in \mathcal{A}_{N,N_0} with the same positions of “0” and hence the same values (19). We may thus insert the factor $\binom{2(N-N_0)}{N-N_0}$ and rather sum over the set \mathcal{B}_{N,N_0} of all strings s over the alphabet $\{0, 1\}$ satisfying $N_0 = N_0(s) = N - N_1(s)/2$. $g_i(s)$ and $G_i(s)$ are defined analogously as before. Then the final result reads

$$\mathcal{Z}_N^\pm(\beta) = \sum_{N_0=0}^N e^{\beta N_0 V} \binom{2(N-N_0)}{N-N_0} \sum_{s \in \mathcal{B}_{N,N_0}} \frac{1}{1 - e^{\beta G_1(s)}} \prod_{n=2}^{|s|} \frac{e^{-\varepsilon^\pm \beta G_n(s)}}{1 - e^{-\beta G_n(s)}}. \quad (20)$$

4 Specific heat and entropy

As usual, we define

$$C_N^\pm(\beta) = \beta^2 \frac{\partial^2}{\partial \beta^2} \ln \mathcal{Z}_N^\pm(\beta). \quad (21)$$

The explicit form would be too complex to be reproduced here. However, using a computer algebra software the specific heat function can be calculated and plotted for small N , say $N \leq 8$, without problems. A typical plot of C_N^- as a function of $\vartheta = \frac{k_B T}{V} = \frac{1}{\beta V}$ is shown in figure 1.

For $\vartheta \rightarrow 0$ the specific heat vanishes exponentially as $\exp\left(-\frac{\hbar\omega}{k_B T}\right)$, which is typical for finite quantum systems. For $\vartheta \approx \vartheta_1$ a first plateau of the height $C_8 = 8$ is reached which is a “preliminary” classical limit of a system of 8 molecules moving in a HO potential, in accordance with the law of Dulong-Petit. In the region $0 < \vartheta < \vartheta_1$ the molecules are practically never broken into parts and behave as pseudo-particles. This is confirmed by a comparison with

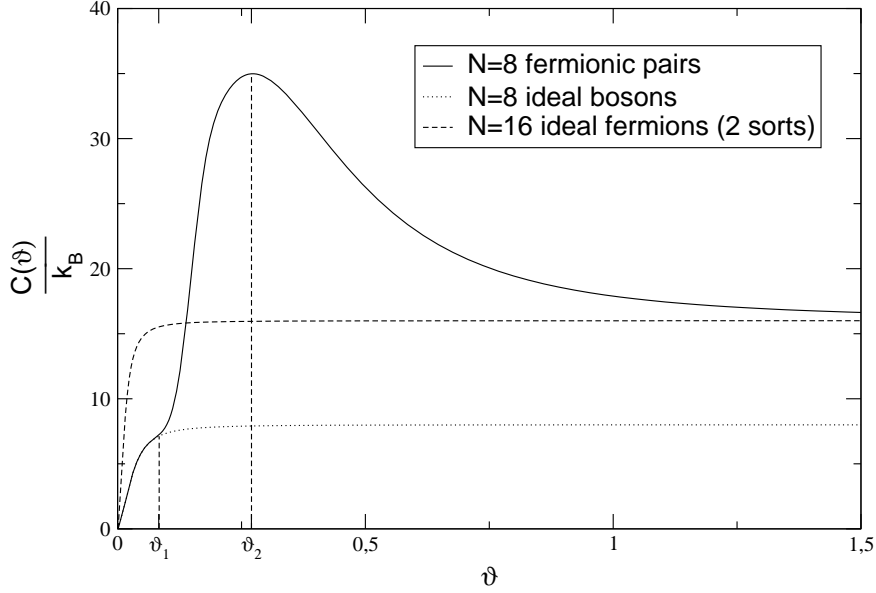


Fig. 1. The specific heat C_8^- as a function of $\vartheta = \frac{1}{\beta V}$ with $V = 100$. This can be compared with the specific heat of system of 8 bosons in a $2\hbar\omega$ -HO potential or of 16 fermions in a $\hbar\omega$ -HO potential.

the specific heat function of 8 bosons (or fermions) in a $2\hbar\omega$ -HO potential, see figure 1. (For the identity $C_N^- = C_N^+$ in 1-dimensional HO potentials see [10].) For $\vartheta > \vartheta_1$ the specific heat increases above its preliminary classical level which indicates a beginning dissociation of the molecules. For $\vartheta \approx \vartheta_3$ the final classical limit $C_6 = 16$ of 16 particles in a HO potential is reached and all molecules are dissociated. But before, at $\vartheta \approx \vartheta_2$, a marked peak of the specific heat occurs which means that at this temperature the maximal portion of any energy supply would be used for the dissociation of the molecules instead of heating up the system. This peak at $\vartheta \approx \vartheta_2$ might indicate a diffuse phase transition, however, the question whether a real phase transition occurs can only be addressed by considering the thermodynamic limit $N \rightarrow \infty$ in the next section and will be answered negatively there.

The entropy function $S(T)$ interpolates between the entropy of 8 ideal bosons for low temperature and of 16 fermions of two different sorts for high temperature, see figure 2.

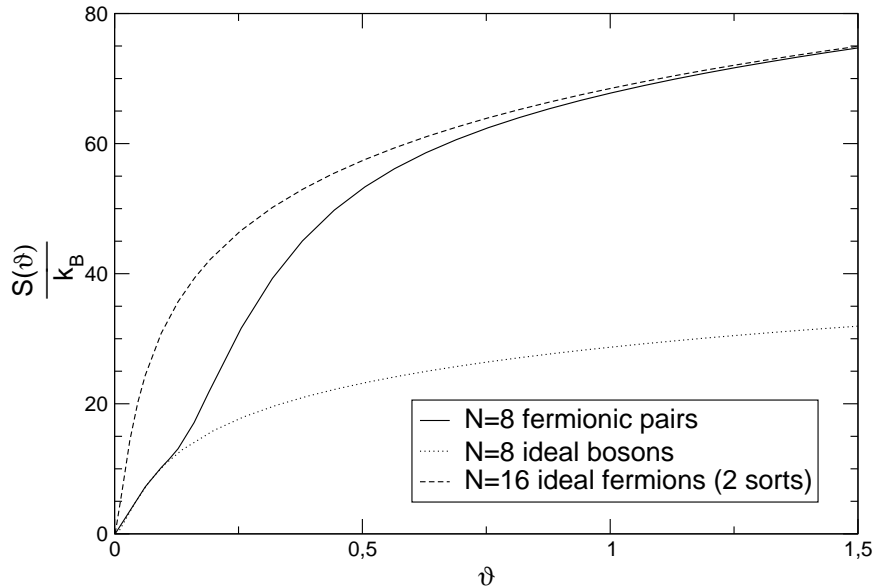


Fig. 2. The entropy S as a function of $\vartheta = \frac{1}{\beta V}$, $V = 100$, for 8 fermion pairs. Compare the corresponding plots of the entropy of an ideal gas of 16 fermions of two different sorts for high temperature and of 8 bosons for low temperature.

5 The thermodynamic limit

It turns out that the explicit solution (20) can also be evaluated in the limit $N \rightarrow \infty$. But we have to perform a simultaneous scaling of the energy level spacing $\hbar\omega$. This is analogous to the usual practice of letting the volume of a system go to infinity in such a way that the particle density remains finite.

To find the correct scaling law in our case, we consider N fermions occupying the first N eigenstates of a HO potential (the “Fermi sea”). Their 1-particle density assumes values which diverge as \sqrt{N} within an interval of length $\sim \sqrt{N}$ in accordance with the total integral of the 1-particle density being N . The typical unit of length of the HO is $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. Hence a scaling $\omega \rightarrow \frac{\omega}{N}$ stretches the system by a factor \sqrt{N} and reduces the 1-particle density to finite values for $N \rightarrow \infty$. For a more detailed account see [11]. In the case of a boson gas we cannot argue in the same way as above, but we will choose the same scaling. In this case the existence of a thermodynamic limit, which is shown below, is the only justification of choosing the scale transformation $\omega \rightarrow \frac{\omega}{N}$.

In practice the considered scaling means that the variable β on the rhs of (20) has to be replaced by $\frac{\beta}{N}$, except for the factor $e^{\beta N_0 V}$ which comes from the interaction, not from the HO potential. Consequently, the corresponding terms can be approximated by their high temperature limit, i. e. by their first order Taylor expansion with respect to β/N . In this limit the difference between the statistics of fermions and bosons disappears.

Hence

$$\frac{e^{-\varepsilon \pm \frac{\beta}{N} G_n(s)}}{1 - e^{-\frac{\beta}{N} G_n(s)}} \simeq \frac{N}{\beta G_n(s)} \quad \beta \text{ fixed, } N \rightarrow \infty \quad (22)$$

and we obtain

$$\mathcal{Z}_N(\beta) \simeq \sum_{N_0=0}^N e^{\beta N_0 V} \binom{2(N - N_0)}{N - N_0} \sum_{s \in \mathcal{B}_{N, N_0}} \prod_{n=1}^{|s|} \frac{N}{\beta G_n} \quad (23)$$

$$= \sum_{N_0=0}^N e^{\beta N_0 V} \binom{2(N - N_0)}{N - N_0} \left(\frac{N}{\beta}\right)^{2N - N_0} \sum_{s \in \mathcal{B}_{N, N_0}} \prod_{n=1}^{|s|} \frac{1}{G_n}. \quad (24)$$

In the next step we calculate the combinatorical factor:

$$\sum_{s \in \mathcal{B}_{N, N_0}} \prod_{n=1}^{|s|} \frac{1}{G_n} = \frac{1}{2_0^N N_0! (2N - 2N_0)!}. \quad (25)$$

In order to prove (25) we first consider an example. Let $N = 6, N_0 = 3, N_1 = 6$ and consider the string $s = (101100111) \in \mathcal{B}_{N, N_0}$. We then have $(g_n) = (121122111)$ and $(G_n) = (12, 11, 9, 8, 7, 5, 3, 2, 1)$. Hence $\prod_n \frac{1}{G_n} = \frac{10 \cdot 6 \cdot 4}{12!}$. Generalizing this example, we see that $\prod_n \frac{1}{G_n}$ can always be written as a fraction with denominator $(2N)!$ and the nominator being a product of N_0 integers n_ν with $1 \leq n_\nu < 2N$ such that their distance is $|n_\nu - n_\mu| \geq 2$ for $\nu \neq \mu$. Conversely, any N_0 integers with these properties occur in the nominator of the terms in the sum $\sum_{s \in \mathcal{B}_{N, N_0}} \prod_{n=1}^{|s|} \frac{1}{G_n}$. We set $2N = M$ and generalize to arbitrary natural numbers M . Let $F(M, N_0)$ be defined as the sum of all products of N_0 -tuples (n_1, \dots, n_{N_0}) with $n_1 < n_2 < \dots < n_{N_0} < M$ and $|n_\nu - n_{\nu+1}| \geq 2$.

If we enlarge M by 1 additionally those N_0 -tuples satisfy the above condition with $n_{N_0} = M$. Hence the following recursion relation holds for F :

$$F(M + 1, N_0) = F(M, N_0) + M F(M - 1, N_0 - 1). \quad (26)$$

Moreover,

$$F(M, N_0) = 0 \text{ for } M < 2N_0 \quad (27)$$

and

$$F(M, 0) = 1 \text{ for all } M. \quad (28)$$

The recursion relation (26) and the initial conditions (27), (28) are satisfied by the following solution

$$F(M, N_0) = \frac{M!}{2^{N_0} N_0! (M - 2N_0)!}, \quad (29)$$

as can be easily confirmed. Setting $M = 2N$ and dividing by $(2N)!$ completes the proof of (25).

Inserting (25) into (24) yields

$$\mathcal{Z}_N(\beta) \simeq \sum_{N_0=0}^N e^{\beta N_0 V} \left(\frac{N}{\beta}\right)^{2N-N_0} \frac{1}{2_0^N N_0! (N - N_0)!^2}. \quad (30)$$

Next we approximate the factorials in (30) by Stirling expressions, e. g. $N_0! \approx N_0^{N_0} e^{-N_0}$, set $N_0 = Nx$ and replace the sum $\sum_{N_0=0}^N \dots$ by an integral $\int_0^1 N dx \dots$. The logarithm of the integrand $I(x)$ reads

$$\ln I(x) = N \{2 - x + V\beta x - x \ln 2 + 2(x - 1) \ln(1 - x) - x \ln x - 2 \ln \beta + x \ln \beta\}. \quad (31)$$

The term in brackets has as a maximum at

$$x_0 = \frac{1}{\beta} e^{-V\beta} \left(1 + \beta e^{V\beta} - \sqrt{1 + 2\beta e^{V\beta}}\right), \quad (32)$$

hence the integrand will be sharply peaked at this maximum if $N \rightarrow \infty$. Thus we can evaluate the integral for $N \rightarrow \infty$ using the saddle point method.

In the last step we remove all factors in the partition function approximation which are not of the form ζ^N , ζ being independent of N . These factors would only give finite N corrections in the thermodynamic functions and have to be neglected in the thermodynamic limit. The final result is

$$\mathcal{Z}_\infty(\beta) = 2^{-2N} \exp \left\{ \frac{N}{\beta} e^{-V\beta} \left(e^{V\beta} \beta - 1 + \sqrt{1 + 2e^{V\beta} \beta} \right) \right\} \left[\frac{\beta}{1 + \sqrt{1 + 2e^{V\beta} \beta}} \right]^{-2N}. \quad (33)$$

The resulting specific heat per molecule will be independent of N :

$$c(\beta) = \frac{\beta^2}{N} \frac{\partial^2}{\partial \beta^2} \ln \mathcal{Z}_\infty(\beta) \quad (34)$$

$$= \frac{2(2(1 + \sqrt{1 + 2e^{V\beta} \beta}) + e^{V\beta} \beta(5 + 2V\beta + V^2 \beta^2 + \sqrt{1 + 2e^{V\beta} \beta}))}{\sqrt{1 + 2e^{V\beta} \beta}(1 + \sqrt{1 + 2e^{V\beta} \beta})^2} \quad (35)$$

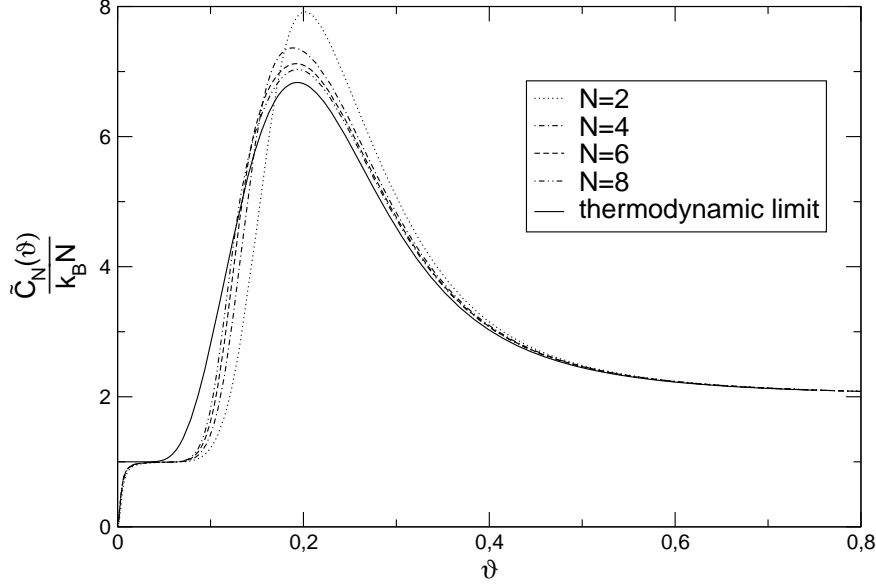


Fig. 3. The specific heat per molecule \tilde{C}_N^-/N as a function of $\vartheta = \frac{1}{\beta V}$ with $V = 100$ for $N = 2, \dots, 8$ which rapidly converge to the thermodynamic limit function $c(\beta)$, except for low temperatures. The tilde in \tilde{C}_N^-/N indicates the scaling $\omega \rightarrow \frac{\omega}{N}$.

The rapid convergence of the finite N specific heat per molecule, scaled with $\omega \rightarrow \frac{\omega}{N}$, to the limit (35) is illustrated in figure 3.

The limits $\lim_{\beta \rightarrow 0} c(\beta) = 2$ and $\lim_{\beta \rightarrow \infty} c(\beta) = 1$ are in accordance with our discussion in section 4.

We define the “degree of dissociation” $\nu(\beta)$ as the thermal expectation value of $1 - \frac{N_0}{N} = 1 - x$, or, equivalently, as

$$\nu(\beta) = 1 - \frac{1}{N\beta} \frac{\partial}{\partial V} \ln \mathcal{Z}_\infty(\beta) \quad (36)$$

and obtain the simple result

$$\nu(\beta) = \frac{2}{1 + \sqrt{1 + 2e^{V\beta}\beta}}. \quad (37)$$

A plot of the degree of dissociation as a function of ϑ , see figure 4 confirms our previous interpretation of the specific heat function.

Since (33) is a smooth function of β there is no phase transition in our model. Instead of a critical temperature at which the molecule phase is changing

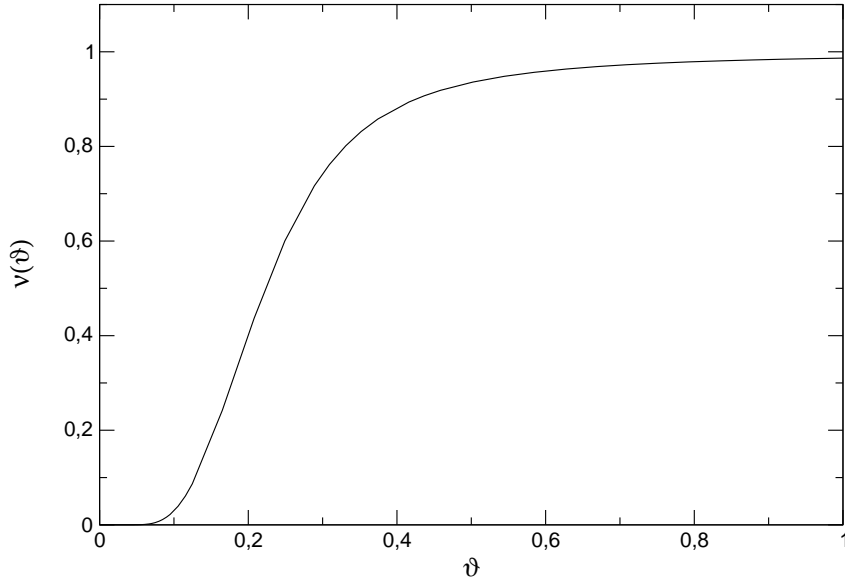


Fig. 4. The degree of dissociation ν as a function of $\vartheta = \frac{1}{\beta V}$, $V = 100$, in the thermodynamic limit.

into the fermi gas phase we have an extended temperature domain where the transition occurs, even in the thermodynamic limit. The reason of this may be that our model is only 1-dimensional and, moreover, is not suited to describe long range order.

6 Summary and outlook

We have studied a two-species quantum gas of $2N$ particles, fermions or bosons, confined in a 1-dimensional HO-potential with a simplified attractive 2-body interaction between particles of different sorts. The canonical partition function and hence the relevant thermodynamical functions of this system can be written in terms of finite sums and products and explicitly calculated by using computer-algebraic means for $N \leq 8$. The specific heat function shows a “preliminary” classical saturation value of $c \sim Nk_B$ for medium temperature, a peak indicating the increasing dissolution of the pairs and a high temperature limit of $c \sim 2Nk_B$. Using the scaling $\omega \rightarrow \frac{\omega}{N}$ we calculated the thermodynamic limit, i. e. the limit of $\frac{1}{N} \ln Z_N^\pm$ and related functions for $N \rightarrow \infty$. In this limit the difference between bosons and fermions vanishes. The peak of the specific heat function remains smooth and no phase transition occurs.

We motivated the construction of our model by considering the problem of pseudo-bosons, that is, the problem, in which sense pairs or, more generally, even numbers of fermions could be considered as bosons. What can be learnt from our model concerning this question? As expected, the behavior of the specific heat function mentioned above deviates from that of an N -particle bose gas, see figure 1, similarly for the entropy function, see figure 2.

However, for low temperatures the difference between bosons and pseudo-bosons is not so marked as we would expect in the general case. This is partly due to the circumstance that the specific heat functions of ideal fermi and bose gases in 1-dimensional HO potentials are the same, see [10]. Only the ground state energies differ by a constant. Also the entropy function $S(T)$ of the pseudo-bose gas is very well approximated by the corresponding bose gas function at low temperatures. We expect that the differences would become more marked in the 3-dimensional HO potential case, where Bose-Einstein condensation occurs. Unfortunately, it is not clear how to generalize the method used in this article to that case, and the problem of pseudo-Bose-Einstein condensation remains unsolved for the present.

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