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Symplectic Quantum Structure

Diploma thesis

– revised version –

by

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For G.,
in gratitude for her moral support
during the crucial weeks.

Contents

List of Figures

1. Introduction

The present diploma thesis is supposed to be a contribution regarding the answer to the question for the disposition – or better the root – of a very basic structure of classical mechanics, the symplectic form. To understand its meaning in mechanics and the necessity for the question proposed, we will start this introduction with a short excursus to the classical theory.

1.1. Some words on the symplectic form in classical mechanics

In the Hamilton formalism of classical mechanics, the 2N-dimensional phase space is the central quantity. It can be formulated interpreting the phase space as a symplectic manifold (M, ω^2) – which is a differential manifold M equipped with a closed, nondegenerate 2-form ω^2 – of dimension $2N$. Together with a real-valued function H on this manifold, called the Hamilton function or Hamiltonian, it constitutes the mechanical system.

By

$$\omega^1_X := \omega^2(\square, X) \tag{1.1}$$

with X in TM, the symplectic form defines an isomorphism between the tangent bundle of M and its space of 1-forms. This is very similar to Riemannian geometry, where the metric tensor provides this map.

We call the inverse of the above isomorphism $I : \Omega^1 M \to TM$. Since functions on M can be viewed as 0-forms and thus, their derivations are 1-forms, it furthermore defines a connection between functions f and vector fields.

We call the vector field $I(df)$ Hamiltonian vector field of the function and denote it by $\mathrm{Ham}\,(f)$ or just X_f in the following. Then

$$df(X) = Xf = \omega^2(X, \mathrm{Ham}\,(f)) \tag{1.2}$$

(Note that the sign may vary between different authors.)

The Poisson bracket of two functions $f, g \in C^\infty(M)$ describing dynamical quantities is defined by

$$\{f, g\} = \omega^2(\mathrm{Ham}\,(f)\,, \mathrm{Ham}\,(g)) = \mathrm{Ham}\,(f)\,g = -\mathrm{Ham}\,(g)\,f. \qquad (1.3)$$

It is not surprising that in classical mechanics especially $I(dH)$ plays an important role. For example, on a flat Euclidean phase space $\mathbb{R}^{2N}$

$$\dot{\mathbf{x}} = I(dH)(\mathbf{x}) \Leftrightarrow \dot{p}_i = -\frac{\partial H}{\partial q^i}, \dot{q}^i = \frac{\partial H}{\partial p_i} \qquad (1.4)$$

where $\mathbf{x} = (p_i, q^i)$ and consequently

$$X_H = (-\frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial p_i}). \qquad (1.5)$$

The Hamiltonian vector field X_H is thus the geometrical representation of the Hamiltonian equations in the phase space. Its integral curves are the solutions of the equations of motion.

A very detailed exposition of the mathematical methods of classical mechanics is given in [Arn88] and [AM78], which have also been used above. We will content ourselves with this introduction and turn to quantum mechanics.

1.2. Quantum mechanics

In quantum physics, different pictures have been developed and are used in different fields of modern physics. The Schrödinger picture is widely used in nuclear and atomic physics, while the Heisenberg picture is the common picture used in quantum field theory. Interestingly – although both pictures strongly inherit the Hamiltonian theory – neither the phase space nor the symplectic form play any role in them. It is essentially replaced by the density matrix.

As a quick answer it might come to one's mind that – due to the uncertainty principle – points in phase space do not make sense in quantum physics since they combine a simultaneous sharp measurement of position and momentum. This is true, but it does only mean that a continuous phase space can only be viewed as a semiclassic limit. It does not forbid to regard a discrete phase space with cells that are large enough not to violate the uncertainty principle, which is a cell size larger than Planck's constant.

[KN91] provides a formulation of a phase space picture of quantum mechanics including a translation of the uncertainty principle into phase space terms. Furthermore, they show different applications of their picture mainly in the field of quantum optics. Although, despite the elaborate formulation, the symplectic form still does not play an important role.

In the thesis on hand, we will examine this basic – but still very poorly known – structure on quantum phase space, starting with results from [KDVM90b] and [Ald01]. We will find that there is a canonical choice for a symplectic structure in the noncommutative differential geometry on the algebra of $N \times N$ matrices, which is closely related to already well-known concepts. We will then find a way from there to the quantum analog of the classical phase space using a special unitary basis, called the Schwinger basis, which we will introduce later on.

In the last part, we will proceed to formulate a connection between the quantum symplectic form derived in the first part and the holonomy and curvature of (principal) fiber bundles.

1.3. Premises

In the above section, it has already become clear that the physical basics of this treatise lie in classical and quantum mechanics. To understand the context, at least the knowledge of standard lectures about classical mechanics and quantum physics should be present. The main emphasis however lies on rather mathematical questions.

The reader's knowledge about the basics of commutative differential geometry (CDG) will be assumed in this diploma thesis, in spite of the fact that it is not part of every physicists course of studies. It is broadly discussed in literature such as [Nak90] or [CBDM82], and would go beyond the scope of this thesis. Instead, we will start directly with a noncommutative version of differential geometry (NDG) by discussing it as an analogon to the common theory.

The second part of this treatise makes extensive use of the theory of fiber bundles. An introduction covering the needed aspects of this theory will be given before we start deriving new results.

Additionally, for all parts of this work, knowledge about the very basic aspects of group theory should be present.

2. A symplectic form in the quantum phase space

2.1. A noncommutative differential geometry

In this section a theory of a noncommutative differential geometry (NDG) of matrix algebras is presented. We will basically rely on the results shown in [KDVM90b] and [Ald01].

In commutative differential geometry we have differential forms that map fields on a differential manifold M into the commutative algebra (with pointwise multiplication) $C^\infty(M)$ of smooth functions on M. In noncommutative differential geometry, these functions are replaced by matrices, which form a noncommutaive algebra. This leads us to the first

Definition 1 *The algebra of smooth functions $C^\infty(M)$ in CDG is replaced by the algebra of complex $N \times N$ matrices (where we should demand $N \geq 2$ for noncommutativity) $M_N(\mathbb{C})$. As in CDG, we identify the space of 0-forms $\Omega^0(M_N(\mathbb{C}))$:*

$$C^\infty(M) = \Omega^0(M) \to M_N(\mathbb{C}) = \Omega^0(M_N(\mathbb{C})) \tag{2.1}$$

Before we consider forms of higher order, we have to define vector fields. In CDG, the Lie algebra of vector fields can be identified with the Lie algebra of derivations Der $(C^\infty(M))$. In NGD we can do just the same. A derivation in the general definition is any unary function D satisfying the Leibniz product law

$$D(MN) = (DM)N + M(DN) \tag{2.2}$$

Let us consider the adjoint action, which is defined by every matrix $A \in M_N(\mathbb{C})$:

$$\begin{aligned} ad : M_N(\mathbb{C}) &\to End(M_N(\mathbb{C})) \\ ad_A := ad(A) : B &\mapsto [A, B] \forall B \in M_N(\mathbb{C}) \end{aligned}$$

Since the commutator obeys

$$[AB, C] = A[B, C] + [A, C]B, \tag{2.3}$$

it obviously provides a derivation. It can be shown, that every derivation on $M_N(\mathbb{C})$ is of this form for a matrix $A \in M_N(\mathbb{C})$ (see [KDVM90b]).

So we have the following

Definition 2 *The Lie algebra of vector fields on a differentiable manifold M is replaced by the Lie algebra of derivations on* $M_N(\mathbb{C})$*:*

$$\mathcal{X}\,(M, TM) = \mathrm{Der}\,(M) \to \mathrm{Der}\,(M_N(\mathbb{C}))$$

A basis for the algebra $M_N(\mathbb{C})$, $\{E_i\}$, provides a basis of $\mathrm{Der}\,(M_N(\mathbb{C}))$, $\{e_i\}$, where $e_k = ad_{E_i}$. Then, an arbitrary derivation X can be written as

$$X = X^i e_i. \tag{2.4}$$

Let us consider forms of higher degrees. Strictly analogue to CDG, we can use the following

Definition 3 (see [Mad99]) *A differential form of order p or just p-form is a p-linear completely antisymmetric map of* $\mathrm{Der}\,(M_N(\mathbb{C}))$ *into* $M_N(\mathbb{C})$.

Since in CDG functions commute, for two forms df, dg

$$df \wedge dg = -dg \wedge df. \tag{2.5}$$

In our noncommutative differential matrix geometry, this is not the case in general:

$$dMdN \neq -dNdM \tag{2.6}$$

Since the product lacks some of the usual properties, we omit the wedge symbol between matrix-valued forms. Nevertheless, the product is still defined by the well known formula for the wedge product between two forms $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^p(M)$ with results in $\Omega^{p+q}(M)$

$$\alpha \wedge \beta(X_1, \ldots, X_{p+q}) = \frac{1}{(p+q)!} \sum_{\{\pi\}} \sigma(\pi)\alpha(X_{\pi(1)}, \ldots, X_{\pi(p)})\beta(X_{\pi(p+1)}, \ldots, X_{\pi(p+q)}) \tag{2.7}$$

where $\sum_{\{\pi\}}$ is the sum over all possible permutations of the indices and $\sigma(\pi)$ is the signature of the permutation π. The only difference is that we have to keep the factors strictly in the order shown.

As outlined before, forms map elements of $\mathrm{Der}\,(M_N(\mathbb{C}))$ into $M_N(\mathbb{C})$. Since we still deal with Lie algebras, the deRham complex can be adopted (see [KDVM90b], [KDVM90a]). For a $p-1$-form α

$$\begin{aligned} d\alpha(X_0, X_1, ..., X_p) = \sum_{0 \leq k \leq p} (-)^k X_k \alpha(X_0, ..., X_{k-1}, X_{k+1}, ..., X_p) + \\ \sum_{0 \leq r < s \leq p} (-)^{r+s} \alpha([X_r, X_s], X_0, ..., X_{r-1}, X_{r+1}, ..., X_{s-1}, X_{s+1}, ..., X_p) \end{aligned} \tag{2.8}$$

applies.

In the special case of $d : \Omega^0 \rightarrow \Omega^1$ for an arbitrary matrix $M \in M_N(\mathbb{C})$ and a derivation $X_A = ad_A$, $A \in M_N(\mathbb{C})$

$$dM(X_A) = X_A M = ad_A M = [A, M] \tag{2.9}$$

follows directly from equation (2.8).

We can now see the deeper reason for equation (2.6):

$$((dM)N)(X_A) = dM(X_A)N = [A, M]N \neq N[A, M] = (NdM)(X_A) \tag{2.10}$$

Nevertheless $d^2 = 0$ is preserved. A 0-form for example gives:

$$\begin{aligned} (d^2M)(X_1, X_2) &= (d(dM))(X_1, X_2) \\ &= X_1(dM)(X_2) - X_2(dM)(X_1) - dM([X_1, X_2]) \\ &= X_1(X_2M) - X_2(X_1M) - [X_1, X_2]M \\ &= (X_1X_2 - X_2X_1)M - [X_1, X_2]M = 0 \end{aligned} \tag{2.11}$$

Beneath the derivation of positive degree n of $\Omega(M_N(\mathbb{C}))$, d^n, an element $X \in \mathrm{Der}\,(M_N(\mathbb{C}))$ defines an antiderivation i_X of degree -1 by

$$(i_X\alpha)(X_1, ...X_p) = \alpha(X, X_1, ..., X_p) \tag{2.12}$$

for any $p+1$-form α. Using Cartan's magic formula

$$L_X = di_X + i_X d \tag{2.13}$$

we can express the Lie derivation (of degree 0), which measures the infinitesimal change of a form under the transformation generated by the field X.

We call an element $\alpha \in \Omega(M_N(\mathbb{C}))$ invariant if it vanishes under the action of L_X for any $X \in \mathrm{Der}\,(M_N(\mathbb{C}))$.

Let us get back to the consideration of bases. Until now we took an arbitrary set of matrices $\{E_k\}$ with

$$[E_i, E_j] = \sum_k C_{ij}^k E_k \tag{2.14}$$

as a basis for the algebra $M_N(\mathbb{C})$. We can lay further restrictions on the elements: While [KDVM90b] demands hermiticity, we will assume them to be unitary, since we have a special unitary basis in mind, the Schwinger basis.

We still lack a basis of $\Omega(M_N(\mathbb{C}))$. An intuitive choice might be the set $\{dE_k\}$. It indeed provides a basis, but since

$$E_k dE_l \neq dE_l E_k \text{ and} \tag{2.15}$$
$$dE_k dE_l \neq -dE_l dE_k, \tag{2.16}$$

this basis is quite difficult to deal with.

Instead, we use the set of 1-forms $\{\theta^k\}$ which are dual to the derivations e_k:

$$\theta^k(e_l) = \delta_l^k \mathbb{1} \tag{2.17}$$

With these forms, one has

$$E_k \theta^l = \theta^l E_k \tag{2.18}$$

and

$$\theta^k \theta^l = \theta^l \theta^k \tag{2.19}$$

as can be easily checked: every basis vector yields 0 or $\mathbb{1}$, which both commute.

With equation (2.14) we get (using Einstein notation here and in the following)

$$dE_k(e_l) = [E_l, E_k] = -C^i_{kl}E_i \tag{2.20}$$

thus using the dual basis

$$dE_k = [E_l, E_k]\theta^l = -C^i_{kl}E_i\theta^l. \tag{2.21}$$

Using the Maurer-Cartan structure equation (see [CBDM82]) the derivation of the dual basis forms yields

$$d\theta^k = -\frac{1}{2}C^k_{lm}\theta^l\theta^m. \tag{2.22}$$

The structure of this matrix differential geometry can be examined much further. [KDVM90b] and [KDVM90a] describe a distinction between real and complex forms, an invariant Riemannian structure, the Hodge theory on $\Omega M_N(\mathbb{C})$, diagonalize the Laplacian for the special case $n = 2$ and use the results to construct an analogue to the Maxwells theory interpreted as a matrix field theory. For our purpose, the presented part is enough and we can proceed to construct a symplectic form.

We recall that a symplectic form is a non-degenerate closed 2-form. To find a canonical choice, consider the 1-form

$$\theta = E_k\theta^k \tag{2.23}$$

It is invariant and we find that it is also independent of the choice of the basis $\{E_k\}$. The first claim has to be checked:

As said before, a 1-form α is called invariant if for all $X \in \mathrm{Der}\,(M_N(\mathbb{C}))$ $L_X\alpha = (i_Xd + di_x)\alpha = 0$. Obviously,

$$i_{e_k}\theta = E_k \tag{2.24}$$

Furthermore

$$\begin{aligned} d\theta &= dE_k\theta^k + E_kd\theta^k \\ &= -C^i_{kl}E_i\theta^l\theta^k + E_k - \frac{1}{2}C^k_l m\theta^l\theta^m \\ &= C^i_{kl}E_i\theta^k\theta^l - \frac{1}{2}C^k_{lm}E_k\theta^l\theta^m \end{aligned}$$

$$= \frac{1}{2} C^k_{lm} E_k \theta^l \theta^m \tag{2.25}$$

Thus

$$d i_{e_j} \theta = dE_j \overset{(2.21)}{=} -C^k_{jm} E_k \theta^m \tag{2.26}$$

and

$$i_{e_j} d\theta = i_{e_j} (\frac{1}{2} C^k lm E_k \theta^l \theta^m) = C^k_{jm} E_k \theta^m, \tag{2.27}$$

which proves the claim.

Indeed, one can show that every invariant element of $\Omega(M_N(\mathbb{C}))$ is a complex multiple of θ (see [KDVM90b]). Thus, the 2-form we are looking for should be closely related to θ. Let us further examine the canonical choice $d\theta$ calculated in equation (2.25) above.

We notice that

$$d\theta(e_i, e_j) = C^k_{ij} E_k = [E_i, E_j] = e_i(E_j). \tag{2.28}$$

We can now define the Hamiltonian vector field $\text{Ham}\,(M)$ associated to a matrix $M \in M_N(\mathbb{C})$ as

$$\text{Ham}\,(M) = ad_M \tag{2.29}$$

and get the properties known from the conventional commutative theory: For any vector field $X \in \text{Der}\,(M_N(\mathbb{C}))$

$$d\theta(X, \text{Ham}\,(M)) = X(M) \tag{2.30}$$

The Poisson bracket working on $M_N(\mathbb{C})$ is then

$$\{M, N\} = d\theta(\text{Ham}\,(M)\,, \text{Ham}\,(N)) = [M, N]. \tag{2.31}$$

We achieve a very well-known structure, if we replace θ by $\hbar\theta$ and ad_X by ad_{iX}. The Poisson bracket is then defined as

$$\{M, N\} = d\theta(\text{Ham}\,(M)\,, \text{Ham}\,(N)) = (i/\hbar)[M, N]. \tag{2.32}$$

Thus, our problem can - as expected - be closely related to the Von Neumann equation, which describes time evolution of the density operator $\hat{\rho}$:

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}[H, \hat{\rho}] \tag{2.33}$$

(see [Sch07]). The RHSs of the two equation above are very similar. The connection between the LHSs is given by the Liouville equation of classical statistical mechanics. It states

$$\frac{\partial \rho}{\partial t} = \{H, \rho\} \tag{2.34}$$

where ρ is the phase space distribution function (see [Sch06]).

We have hinted above, that we have a special unitary basis in mind, which we will investigate further in the context we just derived. In the following section we will have a look at its construction and basic properties.

2.2. The finite dimensional Schwinger basis

In this section we will construct a finite-dimensional unitary operator basis, called the Schwinger basis after Julian Schwinger, who first published these considerations in 1960 ([Sch60b], [Sch60c], [Sch60a]). Additionally, material contained in [Hak98] and [Ald01] has been used in this section.

After the subsections on construction and properties, we will take a short digression on an approach to a solution of the Yang-Baxter equation, which was presented in [Ald01], but could be shown to be unsuccessful.

In the following section we will introduce the Schwinger basis into the results of DKM constituted in the last section.

2.2.1. Construction

Consider an orthonormal basis of a Hilbert space $\mathcal{H}_N$ of finite dimension N

$$\{|u_k\rangle\}_{k\in\mathbb{Z}_N} \tag{2.35}$$

and a unitary operator U. We demand the following properties:

$$\mathcal{U}^N = \mathbb{1} \tag{2.36}$$

$$\mathcal{U}\,|u_k\rangle = |u_{k+1}\rangle \text{ and, since } k\in\mathbb{Z}_N \tag{2.37}$$

$$|u_{k+N}\rangle = |u_k\rangle \tag{2.38}$$

$\mathcal{U}$ can thus be viewed as a kind of translation operator in $\mathcal{H}_N$ and its representation in the $|u_k\rangle$ basis is

$$\mathcal{U} = \sum_{k=0}^{N-1} |u_{k+1}\rangle\,\langle u_k| \tag{2.39}$$

or

$$\mathcal{U} \hat{=} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \tag{2.40}$$

in matrix form.

Now, let $\{|v_k\rangle\}_{k\in\mathbb{Z}_N}$ be the set of normalized eigenvectors of $\mathcal{U}$. It is well known and can be easily derived by calculation that

$$\mathcal{U}\,|v_k\rangle = e^{i\frac{2\pi}{N}k}\,|v_k\rangle \tag{2.41}$$

and

$$|v_k\rangle = \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1} e^{-i\frac{2\pi}{N}kl}\,|u_l\rangle \tag{2.42}$$

up to a global phase factor.

Since these vectors are eigenvectors of a unitary operator to different eigenvalues, this forms another orthonormal basis:

$$\begin{aligned} \langle v_j|v_k\rangle &= N^{-1}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}jm}e^{-i\frac{2\pi}{N}kn}\,\langle u_m|u_n\rangle \\ &= N^{-1}\sum_{m=0}^{N-1} e^{i\frac{2\pi}{N}m(j-k)} = \delta_{jk}, \end{aligned} \tag{2.43}$$

It is the Fourier dual of the $|U_k\rangle$ basis, as it is obvious in equation (2.42):

$$\mathcal{F}\{|u_k\rangle\} = \{|v_k\rangle\} \tag{2.44}$$

$$\mathcal{F}^{-1}\{|v_k\rangle\} = \{|u_k\rangle\} \tag{2.45}$$

where $\mathcal{F}$ is the operator of discrete Fourier transformation as shown in equation (2.42) (see [Ald01] or [LP05]).

Obviously, for this basis we can find a second unitary operator $\mathcal{V}$ which holds

$$\mathcal{V}\left|v_k\right\rangle = \left|v_{k+1}\right\rangle \text{ and, since } k \in \mathbb{Z}_N \tag{2.46}$$

$$\mathcal{V}^N = \mathbb{1}. \tag{2.47}$$

Its $\{|u_k\rangle\}$-representation is

$$\mathcal{V} = \sum_{i=0}^{N-1} \omega^i \left|u_i\right\rangle \left\langle u_i\right| \tag{2.48}$$

where we introduced a shortcut $\omega^k = e^{i\frac{2\pi}{N}k}$ for the kth root of unity. As one can easily check with equation (2.42),

$$\begin{aligned}
\mathcal{V}\left|v_k\right\rangle &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} \omega^m \omega^{kn} \left|u_m\right\rangle \left\langle u_m|u_n\right\rangle \\
&= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \omega^{m(k+1)} \left|u_m\right\rangle \\
&= \left|v_{k+1}\right\rangle
\end{aligned} \tag{2.49}$$

The matrix form with regard to the $|u_k\rangle$-basis obviously is

$$\mathcal{V} \hat{=} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \omega^1 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \omega^{N-2} & 0 \\ 0 & 0 & \vdots & 0 & \omega^{N-1} \end{pmatrix} \tag{2.50}$$

Thus the action of $\mathcal{V}$ on $|u_k\rangle$ is

$$\mathcal{V}\left|u_k\right\rangle = \omega^k \left|u_k\right\rangle \tag{2.51}$$

and the $|u_k\rangle$ are eigenvectors of $\mathcal{V}$.

Let us have a look at the product of these operators:

$$\mathcal{U}\mathcal{V} = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} \omega^n \left|u_{m+1}\right\rangle \left\langle u_m|u_n\right\rangle \left\langle u_n\right|$$

$$= \sum_{m=0}^{N-1} \omega^m \left|u_{m+1}\right\rangle \left\langle u_m\right| \tag{2.52}$$

whereas

$$\begin{aligned}
\mathcal{V}\mathcal{U} &= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \omega^n \left|u_n\right\rangle \left\langle u_n | u_{m+1}\right\rangle \left\langle u_m\right| \\
&= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \omega^{n+1} \left|u_{n+1}\right\rangle \left\langle u_{n+1} | u_{m+1}\right\rangle \left\langle u_m\right| \\
&= \sum_{m=0}^{N-1} \omega^{m+1} \left|u_{m+1}\right\rangle \left\langle u_m\right| \\
&= \omega\mathcal{U}\mathcal{V}
\end{aligned} \tag{2.53}$$

More generally, this means

$$\mathcal{U}^m\mathcal{V}^n = \omega^{mn}\mathcal{V}^n\mathcal{U}^m \tag{2.54}$$

and

$$\begin{aligned}
(\mathcal{U}^m\mathcal{V}^n)^p &= \mathcal{U}^m\mathcal{V}^n \cdot \dots \cdot \mathcal{U}^m\mathcal{V}^n \\
&= \left(\Pi_{k=0}^{p-1}\omega^{kmn}\right) \mathcal{U}^{pm}\mathcal{V}^{pn} \\
&= \omega^{\left(\sum_{k=0}^{p-1} k\right) mn}\mathcal{U}^{pm}\mathcal{V}^{pn} \\
&= \omega^{\frac{p^2-p}{2} mn}\mathcal{U}^{pm}\mathcal{V}^{pn}
\end{aligned} \tag{2.55}$$

In matrix form, each power of $\mathcal{U}$ leads to a shift of the $\mathcal{V}^n$ matrix by one row, while each power of $\mathcal{V}$ directly multiplies the powers of the diagonal elements:

$$\mathcal{U}^m\mathcal{V}^n = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots N-m-4\text{ cols}\cdots & 0 & \omega^{N-m} & 0 & \cdots \\ \vdots & & & & & & \ddots & \ddots & \ddots \\ m-2\text{ rows} & \vdots & \vdots & 0 & \cdots & 0 & & & \\ \vdots & & & \vdots & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & & & & \\ \omega^n & 0 & 0 & 0 & \cdots & & & & \\ 0 & \omega^{2n} & 0 & 0 & \cdots & & & & \\ 0 & 0 & \omega^{3n} & 0 & \cdots & 0 & & & \\ 0 & 0 & 0 & \ddots & \ddots & & & & \\ \vdots & \vdots & \vdots & \ddots & & & & & \end{pmatrix} \tag{2.56}$$

Thus it is easy to see that the set $\{\mathcal{U}^m\mathcal{V}^n\}_{0\leq m,n\leq N-1}$ is linearly independent and has cardinality N^2, thus constituting a basis for the complex vector space $M_N(\mathbb{C})$.

We could use this basis with the considerations of the last chapter now, but first we will slightly modify it to achieve some quite convenient properties. Instead of taking the plain product we add a complex factor $\omega^{mn/2}$. The result is called Schwinger basis and its elements will be denoted as $S_{m,n}$ in the following:

$$S_{m,n} = \omega^{mn/2}\mathcal{U}^m\mathcal{V}^n = e^{\frac{\pi i}{N}mn}\mathcal{U}^m\mathcal{V}^n \tag{2.57}$$

Notice, that we did not define this basis over $\mathbb{Z}_N$ but over the numbers $0 \leq m, n \leq N-1$, since the factor $\omega^{mn/2}$ might not be equal for every member of the equivalence classes defining $\mathbb{Z}_N$, depending on the actual dimension. We will examine this further in the next chapter, were we consider the Schwinger elements to be extended to $\mathbb{Z} \times \mathbb{Z}$

2.2.2. Properties

We first check some very basic properties

$$S_{0,0} = \omega^0\mathcal{U}^0\mathcal{V}^0 = \mathbb{1} \tag{2.58}$$

$$S_{m,n}^{-1} = \omega^{-mn/2}\mathcal{V}^{-n}\mathcal{U}^{-m} = \omega^{mn/2}\mathcal{U}^{-m}\mathcal{V}^{-n} = S_{-m,-n} \tag{2.59}$$
$$= \omega^{-mn/2}\mathcal{V}^{-n}\mathcal{U}^{-m} = \overline{\omega^{mn/2}}\mathcal{V}^{n*}\mathcal{U}^{m*} = S_{m,n}^{*} \tag{2.60}$$

Furthermore the following properties apply:

1. The multiplication of multiple elements is **associative**, since it is only a special case of matrix multiplication:

$$(S_{k,l}S_{m,n})S_{r,s} = S_{k,l}(S_{m,n}S_{r,s}). \tag{2.61}$$

2. It is **quasi-periodic** in each index:

$$S_{N,k} = e^{\frac{\pi i}{N}Nk}\mathcal{U}^N\mathcal{V}^k = \left(e^{\frac{\pi i}{N}N}\right)^k\mathcal{U}^0\mathcal{V}^k = (-1)^k S_{0,k} \tag{2.62}$$
$$S_{k,N} = e^{\frac{\pi i}{N}Nk}\mathcal{U}^k\mathcal{V}^N = \left(e^{\frac{\pi i}{N}N}\right)^k\mathcal{U}^k\mathcal{V}^0 = (-1)^k S_{k,0} \tag{2.63}$$

and especially

$$S_{N,N} = e^{\frac{\pi i}{N}N^2}\mathcal{U}^N\mathcal{V}^N = \left(e^{\frac{\pi i}{N}N}\right)^N \mathbb{1} = (-1)^N S_{0,0}. \tag{2.64}$$

(this leads to an interesting “double covering mod N” of the of the lattice torus $\mathbb{Z}_N \times \mathbb{Z}_N$, which is examined in [AG90]).

3. **Powers** multiply into the indices:

$$\begin{aligned} S_{m,n}^{p} &= \omega^{mn/2^p}\left(\mathcal{U}^m\mathcal{V}^n\right)^p \\ &= \omega^{pmn/2}\omega^{\frac{p^2-p}{2}mn}\mathcal{U}^{pm}\mathcal{V}^{pn} \\ &= \omega^{\frac{pmpn}{2}}\mathcal{U}^{pm}\mathcal{V}^{pn} = S_{pm,pn}. \end{aligned} \tag{2.65}$$

4. $trS_{m,n} = N\delta_{n0}\delta_{n0}$, since every $\mathcal{U}^m\mathcal{V}^n$ except unity is traceless.

Using these properties, it is easy to see that

$$(S_{m,n})^N = S_{Nm,Nn} = S_{-Nm,-Nn} = (-1)^{Nmn}S_{m,n}. \tag{2.66}$$

Let us calculate the product of two basis elements:

$$\begin{aligned}
S_{m,n}S_{k,l} &= \omega^{mn/2}\omega^{kl/2}\mathcal{U}^m\mathcal{V}^n\mathcal{U}^k\mathcal{V}^l \\
&= \omega^{mn/2}\omega^{kl/2}\omega^{nk}\mathcal{U}^{m+k}\mathcal{V}^{n+l} \\
&= \omega^{\frac{mn+ml+nk+kl}{2}+\frac{nk-ml}{2}}\mathcal{U}^{m+k}\mathcal{V}^{n+l} \\
&= \omega^{\frac{nk-ml}{2}}\omega^{\frac{(m+k)(n+l)}{2}}\mathcal{U}^{m+k}\mathcal{V}^{n+l} \\
&= \omega^{\frac{nk-ml}{2}}S_{m+k,n+l} = e^{\frac{i\pi}{N}(nk-ml)}S_{m+k,n+l} = e^{-i\alpha_2((m,n),(k,l))}S_{m+k,n+l} \qquad (2.67)
\end{aligned}$$

where the phase factor contains the function $\alpha_2((m,n),(k,l)) = \frac{\pi}{N}(ml - nk)$ and

$$\begin{aligned}
S_{m,n}S_{k,l} &= e^{-i\alpha_2((m,n),(k,l))}S_{m+k,n+l} \\
&= e^{i\alpha_2((k,l),(m,n))}S_{k+m,l+n} \\
&= e^{2i\alpha_2((k,l),(m,n))}S_{k,l}S_{m,n} \\
&= e^{-2i\alpha_2((m,n),(k,l))}S_{k,l}S_{m,n}. \qquad (2.68)
\end{aligned}$$

α_2 will play an important role in the following chapter.

The action of $S_{m,n}$ on $|u_k\rangle$ is easy to calculate:

$$\begin{aligned}
S_{m,n}|u_k\rangle &= \omega^{mn/2}\mathcal{U}^m\mathcal{V}^n|u_k\rangle \\
&= \omega^{mn/2}\omega^{kn}\mathcal{U}^m|u_k\rangle \\
&= e^{i\frac{\pi}{N}n(2k+m)}|u_{k+m}\rangle \\
&= e^{i\alpha_1(k,(m,n))}|u_{k+m}\rangle \qquad (2.69)
\end{aligned}$$

where we note that the phase $\alpha_1(k,(m,n)) = \frac{\pi}{N}n(2k+m)$ depends on the state this time. [1]

Since every product is linearly dependent on every other product with the same index sum, we can express it in the more general form

$$S_{k,l}S_{r,s} = \sum_{(m,n),(i,j)} B^{(k,l)(r,s)}_{(m,n),(i,j)}S_{i,j}S_{m,n} \qquad (2.70)$$

The solution for B is by far not unique. We chose an ansatz for the solution in which

[1] The 1-form α_1 is closely connected to the 2-form $alpha_2$. See [AG90] and [Ald01] for further examination.

every index combination giving the particular sum is represented, while the rest is 0, since such an addend would be linear independent of the result and had to just be annihilated by another addend of the sum:

$$B^{(k,l)(r,s)}_{(m,n),(i,j)} = \delta^{k+r}_{m+i}\delta^{l+s}_{n+j}\gamma \tag{2.71}$$

To express γ, we consider equations (2.67) and (2.68) again. Since there are N^2 different combinations of indices which add up to (k+r, l+s):

$$\begin{aligned} S_{k,l}S_{r,s} &= e^{-i\alpha_2((k,l),(r,s))}S_{k+r,l+s} \\ &= \frac{1}{N^2}\sum_{(m,n),(i,j)} \delta^{k+r}_{m+i}\delta^{l+s}_{n+j}e^{-i\alpha_2((k,l),(r,s))}e^{i\alpha_2((i,j),(m,n))}S_{i,j}S_{m,n} \end{aligned} \tag{2.72}$$

Even this solution is not unique. We could add arbitrary weightings to each addend, but since there is no superior distribution, we will stay with weighting every term equally.

Consequently $B^{(k,l)(r,s)}_{(m,n),(i,j)}$ can be chosen as

$$B^{(k,l)(r,s)}_{(m,n),(i,j)} = \frac{1}{N^2}\delta^{k+r}_{m+i}\delta^{l+s}_{n+j}e^{i(\alpha_2((i,j),(m,n))-\alpha_2((k,l),(r,s)))} \tag{2.73}$$

We will examine this representation in the subsection after the next one.

2.2.3. The group structure

Let us have a look at the group structure of the Schwinger basis. As shown in equation (2.67), using the common matrix multiplication, the basis elements do not form a group since multiplications of two elements generate phase factors. We can close the group under the group operation "matrix multiplication", if we extend it by the phase factors. The group is then generated by the set $\{\omega, S_{1,0}, S_{0,1}\}$. A group element has the general form $S_{m,n}\omega^p$ with $m,n \in \mathbb{Z}_N$, $q \in \frac{\mathbb{Z}_{2N}}{2}$. Consequently, the group has $2N^3$ elements.

Consider the multiplication behavior:

$$
\begin{aligned}
(S_{m,n}\omega^p)\cdot(S_{r,s}\omega^q) &= S_{m+r,n+s}\omega^{\frac{nr-ms}{2}}\omega^{p+q} \\
&= S_{m+r,n+s}\omega^{\frac{nr-ms}{2}+p+q}
\end{aligned}
\tag{2.74}
$$

or, just for the indices

$$
(m,n,p)\star(r,s,q) = (m+r, s+n, p+q+\frac{nr-ms}{2}). \tag{2.75}
$$

This is the defining relation of a Heisenberg group on a symplectic manifold, where basically α_2 is the symplectic form (see [CDPT07]). If we resubstitute the Schwinger elements with the Weyl operators $\mathcal{U}$,$\mathcal{V}$, we just get the Weyl realization of the Heisenberg group described in [Wey50].

2.2.4. Schwinger, braiding and the Yang-Baxter equation

An introduction to braiding

Since here lies the initial target of this work we will take a brief side trip to braiding. It has been discussed at length in [Bir75]. Here, just a short introduction will be given. Braiding can be seen as a generalization of permutation. Indeed, the permutation groups are the special braiding groups, in which the action on two elements is its own inverse. In general, this is not true.

Consider for example three balls: We can take the first ball and put it in the second position and vice versa. We will call this action σ_1, while σ_2 means exchanging the second with the third one. Performing the very same action again leads to the initial state.

Now imagine three strings connecting the balls to fixed points. If you perform σ_1 twice now, the result will be two twisted strings. They are "braided". Since we can always twist the strings further, the group has infinite members, e.g. $\{\sigma_1^n | n \in \mathbb{Z}\}$ where $\sigma^{-n} := \left(\sigma^{-1}\right)^n$ for the smallest non-trivial case with two elements.

Let us check for identities. Obviously, braid operations commute, if the sets of elements they affect are disjoint. For simple elements (as we call those that just effect two nearby elements and can be expressed as σ_i) this means

$$\sigma_i \sigma_{i+k} = \sigma_{i+k} \sigma_i \text{ for } k > 1 \tag{2.76}$$

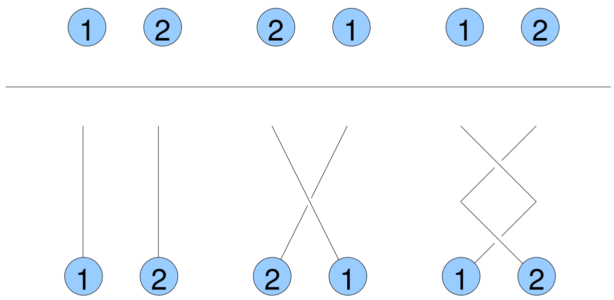

Figure 2.1.: Permutation and braiding of two elements

Another identity is

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{2.77}$$

as can be easily checked, see figure 2.2.

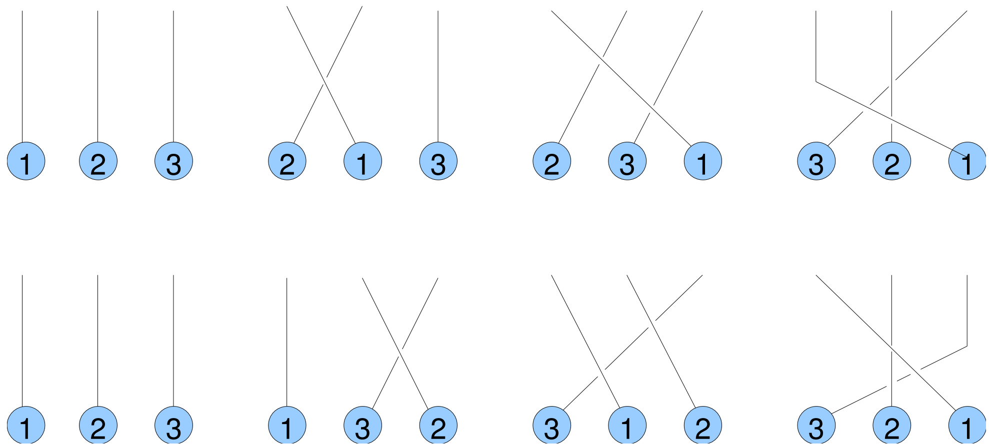

Figure 2.2.: Identity at braiding three elements

The connection to the Yang-Baxter equation

We will now demonstrate how braiding is connected to the Yang-Baxter equation. Consider an N-dimensional Hilbert space $\mathcal{H}_N$ with an orthonormal basis $\{|n\rangle\}_{n\in\mathbb{Z}_n}$ and construct a triple direct product space spanned by $\{|ijk\rangle\}_{i,j,k\in\mathbb{Z}_n}$. A simple braid operator in this space, i.e. an operator braiding two elements of the direct sum, has the form

$$\hat{B}_1 \otimes \hat{B}_2 \otimes \mathbb{1} =: \hat{B}_{12} = \sigma_1 \text{ resp. } \hat{B}_1 \otimes \mathbb{1} \otimes \hat{B}_2 =: \hat{B}_{13} \text{ resp. } \mathbb{1} \otimes \hat{B}_1 \otimes \hat{B}_2 =: \hat{B}_{23} = \sigma_2 \tag{2.78}$$

The elements of these hypermatrices can be written as

$$\langle ijk|\hat{B}_{12}|rst\rangle = \langle ijk|\hat{B}_1 \otimes \hat{B}_2 \otimes \mathbb{1}|rst\rangle = \langle i|\hat{B}_1|r\rangle \langle j|\hat{B}_2|s\rangle \langle k|t\rangle = B^{ij}_{rs}\delta^k_t \tag{2.79}$$

$$\langle ijk|\hat{B}_{13}|rst\rangle = \langle ijk|\hat{B}_1 \otimes \mathbb{1} \otimes \hat{B}_2|rst\rangle = \langle i|\hat{B}_1|r\rangle \langle j|s\rangle \langle k|\hat{B}_2|t\rangle = B^{ik}_{rt}\delta^j_s \tag{2.80}$$

$$\langle ijk|\hat{B}_{23}|rst\rangle = \langle ijk|\mathbb{1} \otimes \hat{B}_1 \otimes \hat{B}_2|rst\rangle = \langle i|r\rangle \langle j|\hat{B}_1|s\rangle \langle k|\hat{B}_2|t\rangle = B^{jk}_{st}\delta^i_r \tag{2.81}$$

where B^{ij}_{rs} are the elements of the hypermatrix $\hat{B} = \hat{B}_1 \otimes \hat{B}_2$.

We can now introduce the identity shown in equation (2.77):

$$\begin{aligned}\langle ijk|\sigma_1\sigma_2\sigma_1|rst\rangle &= \langle ijk|\hat{B}_{12}\hat{B}_{23}\hat{B}_{12}|rst\rangle \\ &= \langle ijk|\hat{B}_{12}|lmn\rangle \langle lmn|\hat{B}_{23}|opq\rangle \langle opq|\hat{B}_{12}|rst\rangle \\ &= B^{ij}_{lm}\delta^k_n B^{mn}_{pq}\delta^l_o B^{op}_{rs}\delta^q_t \\ &= B^{ij}_{lm}B^{mk}_{pt}B^{lp}_{rs}\end{aligned} \tag{2.82}$$

which equals

$$\begin{aligned}\langle ijk|\sigma_2\sigma_1\sigma_2|rst\rangle &= \langle ijk|\hat{B}_{32}\hat{B}_{12}\hat{B}_{23}|rst\rangle \\ &= \langle ijk|\hat{B}_{23}|lmn\rangle \langle lmn|\hat{B}_{12}|opq\rangle \langle opq|\hat{B}_{23}|rst\rangle \\ &= B^{jk}_{mn}\delta^i_l B^{lm}_{op}\delta^n_q B^{pq}_{st}\delta^o_r \\ &= B^{jk}_{mn}B^{im}_{rp}B^{pn}_{st}\end{aligned} \tag{2.83}$$

(Einstein notation applies here and in the following).

Thus, we have

$$B^{ij}_{lm}B^{mk}_{pt}B^{lp}_{rs} = B^{jk}_{mn}B^{im}_{rp}B^{pn}_{st} \tag{2.84}$$

If a hypermatrix $\hat{B}$ satisfies this equation, the matrix $\hat{R}$ derived from $\hat{B}$ by permutation of indices

$$R^{ij}_{lm} = B^{ji}_{lm} \tag{2.85}$$

satisfied

$$R^{ji}_{lm}R^{km}_{pt}R^{pl}_{rs} = R^{kj}_{mn}R^{mi}_{rp}R^{np}_{st} \tag{2.86}$$

which is just the Yang-Baxter-equation.

An unsuccessful approach to a non-trivial solution

In equation (2.70) we found a general expression for the product of two Schwinger basis elements. We can perform some kind of braiding by substituting the elements of a triple product in different orders. Consider the product $S_{i,j}S_{k,l}S_{m,n}$. Since it is associative

$$(S_{\alpha,\beta}S_{\gamma,\delta})S_{\mu,\nu} = S_{\alpha,\beta}(S_{\gamma,\delta}S_{\mu,\nu}). \tag{2.87}$$

Let us calculate the LHS (Einstein notation applies for Latin indices):

$$\begin{aligned}(S_{\alpha,\beta}S_{\gamma,\delta})S_{\mu,\nu} &= B^{(\alpha,\beta)(\gamma,\delta)}_{(c,d)(a,b)}S_{a,b}(S_{c,d}S_{\mu,\nu})\\ &= B^{(\alpha,\beta)(\gamma,\delta)}_{(c,d)(a,b)}B^{(c,d)(\mu,\nu)}_{(g,h)(e,f)}(S_{a,b}S_{e,f})S_{g,h}\\ &= B^{(\alpha,\beta)(\gamma,\delta)}_{(c,d)(a,b)}B^{(c,d)(\mu,\nu)}_{(g,h)(e,f)}B^{(a,b)(e,f)}_{(k,l)(i,j)}S_{i,j}S_{k,l}S_{g,h}\end{aligned} \tag{2.88}$$

The RHS gives

$$\begin{aligned}S_{\alpha,\beta}(S_{\gamma,\delta}S_{\mu,\nu}) &= B^{(\gamma,\delta)(\mu,\nu)}_{(c,d)(a,b)}(S_{\alpha,\beta}S_{a,b})S_{c,d}\\ &= B^{(\gamma,\delta)(\mu,\nu)}_{(c,d)(a,b)}B^{(\alpha,\beta)(a,b)}_{(g,h)(e,f)}S_{e,f}(S_{g,h}S_{c,d})\\ &= B^{(\gamma,\delta)(\mu,\nu)}_{(c,d)(a,b)}B^{(\alpha,\beta)(a,b)}_{(g,h)(e,f)}B^{(g,h)(c,d)}_{(k,l)(i,j)}S_{e,f}S_{i,j}S_{k,l}.\end{aligned} \tag{2.89}$$

Thus, after matching the indices,

$$B^{(\alpha,\beta)(\gamma,\delta)}_{(c,d)(a,b)}B^{(c,d)(\mu,\nu)}_{(q,r)(e,f)}B^{(a,b)(e,f)}_{(m,n)(o,p)}S_{m,n}S_{o,p}S_{q,r} = B^{(\gamma,\delta)(\mu,\nu)}_{(c,d)(a,b)}B^{(\alpha,\beta)(a,b)}_{(g,h)(m,n)}B^{(g,h)(c,d)}_{(q,r)(o,p)}S_{m,n}S_{o,p}S_{q,r} \tag{2.90}$$

In each nontrivial addend in the sum, $(\alpha+\gamma+\mu, \beta+\delta+\nu) = (m+o+q, n+p+e)$, thus they are pairwise linearly dependent. One might accordingly come to the conclusion that

$$B^{(\alpha,\beta)(\gamma,\delta)}_{(c,d)(a,b)}B^{(c,d)(\mu,\nu)}_{(q,r)(e,f)}B^{(a,b)(e,f)}_{(m,n)(o,p)} = B^{(\gamma,\delta)(\mu,\nu)}_{(c,d)(a,b)}B^{(\alpha,\beta)(a,b)}_{(g,h)(m,n)}B^{(g,h)(c,d)}_{(q,r)(o,p)} \tag{2.91}$$

which is just the braid equation (2.84) with double indices. As [Ald01] suggests, this would lead to a non-trivial solution of the Yang-Baxter equation through equation (2.85).

Unfortunately this conclusion offers some severe problems. Although all $S_{m,n}S_{o,p}S_{q,r}$ products are linearly dependent in the sum, they are by no means equal. Indeed equation (2.68) shows us, that we get an additional phase factor:

$$\begin{aligned} S_{m,n}S_{o,p}S_{q,r} &= e^{-i\alpha_2((m,n),(o,p))}S_{m+o,n+p}S_{q,r} \\ &= e^{-i\alpha_2((m,n),(o,p))}e^{-i\alpha_2((m+o,n+p),(q,r))}S_{m+o+q,n+p+r} \\ &= e^{-i\frac{\pi}{N}(mp-np)}e^{-i\frac{\pi}{N}(mr+or-nq-pq)}S_{m+o+q,n+p+r} \\ &= e^{-i\frac{\pi}{N}mp}e^{i\frac{\pi}{N}np}e^{-i\frac{\pi}{N}mr}e^{-i\frac{\pi}{N}or}e^{i\frac{\pi}{N}nq}e^{i\frac{\pi}{N}pq}S_{m+o+q,n+p+r} \end{aligned} \tag{2.92}$$

The last line is the maximal factorization of the exponential functions. Regarding the LHS of equation (2.91), for example the third factor here carries indices which do not appear in a single B-factor. Thus, the phase factors can not be reasonably distributed to the $B^{(a,b)(e,f)}_{(m,n),(o,p)}$ to preserve the form of the Yang-Baxter equation. Notice that this conclusion is independent of the ansatz in equation (2.71).

Neverless, we will check if equation (2.91) might still apply for our ansatz without being related to the Yang-Baxter equation, altough it became quite improbable. The term requires further examination. Since an analytic treatment is a quite cumbersome affair due to the large amount of indices, I started with a computer algebraic check.

In the appendix a Mathematica program including the output of a 1000-values run is presented. Another run over 10000 random combinations of indices has been performed. The short version of the result is as follows:

Trivially satisfied	9582
Non-trivially satisfied	114
Not satisfied	304

The large amount of combinations satisfying the equation trivially, i.e. giving zero on both sides, can be easily explained by the Kronecker deltas in the B-term.

The non-trivial results however show that they just do not satisfy the equation. Since there is no obvious way to improve the expression, the approach presented in [Ald01] can be assumed to have failed. The braid equation does not sufficiently describe the interchange relation of the Schwinger basis elements, since the contraction of a product to a single element yields additional phase factors.

Thus a connection to the Yang-Baxter equation could not be extracted. Only if multiplication leads to addition of indices without yielding phase factors, the connection could be made. Unfortunately, this would automatically lead to commutativity due to the commutativity of the addition of indices and the result would be trivial.

2.2.5. Schwinger matrices as a basis of $M_N(\mathbb{C})$

Let us consider the properties of the Schwinger matrices as a basis of $M_N(\mathbb{C})$. We can express every element $\hat{A} \in M_N(\mathbb{C})$ by

$$\hat{A} = \frac{1}{N} A^{(m,n)} S_{(m,n)} \tag{2.93}$$

where the coefficients $A^{(m,n)}$ are complex numbers. The factor N^{-1} allows us to write them as

$$A^{(m,n)} = Tr\left[S^*_{(m,n)} A\right] \tag{2.94}$$

since

$$Tr\left[S^*_{(m,n)} S_{(r,s)}\right] = N\delta_{m,r}\delta_{n,s} \tag{2.95}$$

using property 4 in subsection 2.2.2 and equations (2.60) and (2.67).

Along the way, we notice that the trace defines an interior product on the operator algebra, in which the Schwinger basis elements are orthonormal

$$\langle \hat{A}, \hat{B} \rangle = N^{-1} Tr\left[B^* A\right] \tag{2.96}$$

$$\langle S^*_{(m,n)} S_{(r,s)} \rangle = \delta_{m,r}\delta_{n,s}. \tag{2.97}$$

We will now examine, how a product of two matrices appears in this representation:

$$\hat{A}\hat{B} = \left(N^{-1} \sum_{(m,n)} A^{(m,n)} S_{(m,n)}\right)\left(N^{-1} \sum_{(r,s)} B^{(r,s)} S_{(r,s)}\right) \tag{2.98}$$

We can use equation (2.67) to derive the coefficients of the product matrix

$$\begin{aligned}
\hat{A}\hat{B} &= N^{-2} \sum_{(m,n),(r,s)} A^{(m,n)} B^{(r,s)} e^{i\alpha_2((r,s),(m,n))} S_{(m+r,n+s)} \\
&= N^{-1} \sum_{(r,s)} \left\{ N^{-1} \sum_{(m,n)} A^{(m,n)} B^{(r,s)} e^{i\alpha_2((r,s),(m,n))} \right\} S_{(m+r,n+s)} \\
&= N^{-1} \sum_{(r,s)} \left\{ N^{-1} \sum_{(m,n)} A^{(m,n)} B^{(r-m,s-n)} e^{i\alpha_2((r-m,s-n),(m,n))} \right\} S_{(r,s)} \\
&= N^{-1} \sum_{(r,s)} \left\{ N^{-1} \sum_{(m,n)} A^{(m,n)} B^{(r-m,s-n)} e^{i\alpha_2((r,s),(m,n))} \right\} S_{(r,s)} \qquad (2.99)
\end{aligned}$$

Thus

$$(AB)^{(\mu,\nu)} = N^{-1} \sum_{(r,s)} A^{(r,s)} B^{(\mu-r,\nu-s)} e^{i\alpha_2((\mu,\nu),(r,s))} \qquad (2.100)$$

This expression has the general form of a convolution, but with an additional phase factor. Such an expression is called "twisted convolution". It is "twisted" since, in contrast to the normal convolution, it is not commutative in our case. With the expression above, we moved the noncommutativity from the operator algebra into the algebra of complex coefficients equipped with the product of the twisted convolution.

It is now easy to transfer these considerations to commutators. We first calculate the commutator of two basis elements:

$$\begin{aligned}
[S_{(m,n)}, S_{(r,s)}] &= S_{(m,n)} S_{(r,s)} - S_{(r,s)} S_{(m,n)} \\
&= e^{i\alpha_2((r,s),(m,n))} S_{(m+r,n+s)} - e^{i\alpha_2((m,n),(r,s))} S_{(m+r,n+s)} \\
&= \left(e^{i\alpha_2((r,s),(m,n))} - e^{-i\alpha_2((r,s),(m,n))} \right) S_{(m+r,n+s)} \\
&= 2i \sin\left(\alpha_2((r,s),(m,n))\right) S_{(m+r,n+s)} \qquad (2.101)
\end{aligned}$$

which yields the structure coefficients

$$C^{(i,j)}_{(m,n)(r,s)} = -2i \sin\left(\alpha_2((m,n),(r,s))\right) \delta^{(i,j)}_{(m+r,n+s)}. \qquad (2.102)$$

Thus for two general matrices,

$$
\begin{aligned}
[\hat{A},\hat{B}] &= N^{-2} \sum_{(m,n),(r,s)} A^{(m,n)} B^{(r,s)} 2i \sin\left(\alpha_2((r,s),(m,n))\right) S_{(m+r,n+s)} \\
&= N^{-1} \sum_{(r,s)} \left\{ N^{-1} \sum_{(m,n)} A^{(m,n)} B^{(r-m,s-n)} 2i \sin\left(\alpha_2((r,s),(m,n))\right) \right\} S_{(r,s)}
\end{aligned}
\tag{2.103}
$$

or

$$
[\hat{A},\hat{B}]^{(\mu,\nu)} = N^{-1} \sum_{(r,s)} A^{(r,s)} B^{(\mu-r,\nu-s)} 2i \sin\left(\alpha_2((\mu,\nu),(r,s))\right) \tag{2.104}
$$

applies.

We would like to have a metric to achieve a “covariant” version of the “contravariant” $S_{\mathbf{m}}$. In equation (2.102) we found the structure coefficients of the Lie algebra in the Schwinger basis. They allow us to calculate the Killing form, which is defined (see [Mad99] or for a more comprehensive treatment [NS82], [Har96] and [Sen99]) as

$$
k_{\mathbf{mn}} = \frac{1}{2N^2} \sum_{\mathbf{rk}} C^{\mathbf{r}}_{\mathbf{mk}} C^{\mathbf{k}}_{\mathbf{nr}} \tag{2.105}
$$

where we have – for the sake of simplicity – chosen another notation by combining the two indices into a multi-index $\mathbf{i} = (i_1, i_2)$. Then $\sum_{\mathbf{i}} \equiv \sum_{i_1=0}^{N-1} \sum_{i_2=0}^{N-1}$ and $\delta^{\mathbf{i}}_{\mathbf{j}} \equiv \delta^{i_1}_{j_1} \delta^{i_2}_{j_2}$.

This notation is equivalent since it can be mapped to N^2 single indices e.g. by

$$
\mathbb{Z}_N \times \mathbb{Z}_N \ni (m_1, m_2) \longmapsto (m_1 \cdot N + m_2) \in \mathbb{Z}_{N^2}. \tag{2.106}
$$

Using equation (2.102) we get

$$
\begin{aligned}
k_{\mathbf{ab}} &= \frac{-4}{2N^2} \sum_{\mathbf{cd}} C^{\mathbf{c}}_{\mathbf{ad}} C^{\mathbf{d}}_{\mathbf{bc}} \\
&= \frac{-4}{2N^2} \sum_{\mathbf{cd}} \sin\left(\alpha_2(\mathbf{a},\mathbf{d})\right) \delta^{\mathbf{c}}_{\mathbf{a}+\mathbf{d}} \sin\left(\alpha_2(\mathbf{b},\mathbf{c})\right) \delta^{\mathbf{d}}_{\mathbf{b}+\mathbf{c}} \\
&= \frac{1}{2N^2} \sum_{\mathbf{cd}} \left(e^{i\alpha_2(\mathbf{a},\mathbf{d})} - e^{-i\alpha_2(\mathbf{a},\mathbf{d})}\right) \left(e^{i\alpha_2(\mathbf{b},\mathbf{c})} - e^{-i\alpha_2(\mathbf{b},\mathbf{c})}\right) \delta^{\mathbf{c}}_{\mathbf{a}+\mathbf{d}} \delta^{\mathbf{d}}_{\mathbf{b}+\mathbf{c}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2N^2}\sum_{\mathbf{cd}}(e^{i\alpha_2(\mathbf{a},\mathbf{d})+i\alpha_2(\mathbf{b},\mathbf{c})} - e^{i\alpha_2(\mathbf{a},\mathbf{d})-i\alpha_2(\mathbf{b},\mathbf{c})} - e^{-i\alpha_2(\mathbf{a},\mathbf{d})+i\alpha_2(\mathbf{b},\mathbf{c})} \\
&\quad + e^{-i\alpha_2(\mathbf{a},\mathbf{d})-i\alpha_2(\mathbf{b},\mathbf{c})})\delta^{\mathbf{c}}_{\mathbf{a}+\mathbf{d}}\delta^{\mathbf{d}}_{\mathbf{b}+\mathbf{c}} \\
&= \frac{1}{2N^2}\sum_{\mathbf{d}}(e^{i\alpha_2(\mathbf{a},\mathbf{d})+i\alpha_2(\mathbf{b},\mathbf{a}+\mathbf{d})} - e^{i\alpha_2(\mathbf{a},\mathbf{d})-i\alpha_2(\mathbf{b},\mathbf{a}+\mathbf{d})} - e^{-i\alpha_2(\mathbf{a},\mathbf{d})+i\alpha_2(\mathbf{b},\mathbf{a}+\mathbf{d})} \\
&\quad + e^{-i\alpha_2(\mathbf{a},\mathbf{d})-i\alpha_2(\mathbf{b},\mathbf{a}+\mathbf{d})})\delta^{\mathbf{d}}_{\mathbf{b}+\mathbf{a}+\mathbf{d}} \\
&= \frac{1}{2N^2}(e^{i\alpha_2(\mathbf{b},\mathbf{a})}\sum_{\mathbf{d}}e^{-i\alpha_2(\mathbf{d},\mathbf{a}+\mathbf{b})} \\
&\quad - e^{-i\alpha_2(\mathbf{b},\mathbf{a})}\sum_{\mathbf{d}}e^{-i\alpha_2(\mathbf{d},\mathbf{a}-\mathbf{b})} - e^{i\alpha_2(\mathbf{b},\mathbf{a})}\sum_{\mathbf{d}}e^{-i\alpha_2(\mathbf{d},\mathbf{b}-\mathbf{a})} \\
&\quad + e^{-i\alpha_2(\mathbf{b},\mathbf{a})}\sum_{\mathbf{d}}e^{-i\alpha_2(\mathbf{d},-\mathbf{b}-\mathbf{a})})\delta_{\mathbf{a},-\mathbf{b}} \\
&= \frac{1}{2N^2}\delta_{\mathbf{a},-\mathbf{b}}\left(e^{i\alpha_2(\mathbf{b},\mathbf{a})}N^2\delta_{\mathbf{a},-\mathbf{b}} - e^{i\alpha_2(\mathbf{b},\mathbf{a})}N^2\delta_{\mathbf{a},\mathbf{b}} - e^{i\alpha_2(\mathbf{b},\mathbf{a})}N^2\delta_{\mathbf{a},\mathbf{b}} + e^{i\alpha_2(\mathbf{b},\mathbf{a})}N^2\delta_{\mathbf{a},-\mathbf{b}}\right) \\
&= \frac{1}{2}\delta_{\mathbf{a},-\mathbf{b}}\left(e^{i\alpha_2(\mathbf{b},\mathbf{a})} + e^{-i\alpha_2(\mathbf{b},\mathbf{a})}\right) - \frac{1}{2}\delta_{\mathbf{a},-\mathbf{b}}\delta_{\mathbf{a},\mathbf{b}}\left(e^{i\alpha_2(\mathbf{b},\mathbf{a})} + e^{-i\alpha_2(\mathbf{b},\mathbf{a})}\right) \\
&= \cos\left(\alpha_2(\mathbf{b},\mathbf{a})\right)\cdot\left(\delta_{\mathbf{a},-\mathbf{b}} - \delta_{\mathbf{a},-\mathbf{b}}\delta_{\mathbf{a},\mathbf{b}}\right) \qquad (2.107)
\end{aligned}
$$

This is a quite unpleasant affair, since besides the phase factors, in even dimensions there exist elements that can satisfy both $\delta_{\mathbf{a},\mathbf{b}}$ and $\delta_{\mathbf{a},-\mathbf{b}}$. We simplify this metric by omitting the phase factors and the second Kronecker delta (the result being actually proposed as the Killing metric in [Ald01]). This metric has some quite useful properties:

$$
S^{\mathbf{m}} = k^{\mathbf{mn}}S_{\mathbf{n}} = \delta_{\mathbf{m},-\mathbf{n}}S_{\mathbf{n}} = S_{-\mathbf{m}} = S_{\mathbf{m}}^{-1} = S^{*}_{-\mathbf{m}} \qquad (2.108)
$$

The commutator is just the symplectic form which we derived in section 2.1. In the following section we will finally combine the present results and examine this symplectic structure using the Schwinger elements as a basis system.

2.3. Non-commutative differential geometry using the Schwinger basis

2.3.1. The symplectic form and the Schwinger basis

While we used an arbitrary basis for the algebra $M_N(\mathbb{C})$ in section 2.1, we will reconsider the results using the Schwinger basis in this section.

Equation (2.14) now becomes

$$[S_{(m,n)}, S_{(r,s)}] = 2i \sin(\alpha_2((r,s),(m,n)))S_{(m+r,n+s)} \tag{2.109}$$

as shown in the last section. Thus, the structure coefficients are

$$C^{(i,j)}_{(m,n)(r,s)} = 2i \sin(\alpha_2((r,s),(m,n)))\delta^i_{m+r}\delta^j_{n+s} \tag{2.110}$$

In the following, we will again use the double-index notation introduced in the previous section for better survey and to preserve the notation used in section 2.1. We keep in mind that what was N in that section is N^2 here.

In this notation, equation 2.110 has the form

$$C^{\mathbf{i}}_{\mathbf{mr}} = 2i \sin(\alpha_2(\mathbf{r},\mathbf{m}))\delta^{\mathbf{i}}_{\mathbf{m+r}}. \tag{2.111}$$

Next we define the generators of the Lie algebra of derivations as $\{e_{\mathbf{i}} = ad(iS_{\mathbf{i}}) = ad_{iS_{\mathbf{i}}} | \mathbf{i} = (i_1.i_2) \neq (0,0)\}$. We had to exclude $S_{\mathbf{0}} = \mathbb{1}$, since it is an element of the center $\mathcal{Z}$ of the algebra and is thus mapped to the $\hat{0}$ operator by the adjoint representation. The center is trivial in our case, this is the only index we have to exclude.

Consider now the commutator $[e_{\mathbf{i}}, e_{\mathbf{j}}]$:

$$\begin{aligned}[e_{\mathbf{i}}, e_{\mathbf{j}}]S_{\mathbf{k}} &= [ad_{iS_{\mathbf{i}}}, ad_{iS_{\mathbf{j}}}](S_{\mathbf{k}}) \\ &= ad_{[iS_{\mathbf{i}}, iS_{\mathbf{j}}]}(S_{\mathbf{k}}) \\ &= -[[S_{\mathbf{i}}, S_{\mathbf{j}}], S_{\mathbf{k}}]\end{aligned}$$

$$
\begin{aligned}
&= 2i \sin(\alpha_2(\mathbf{i},\mathbf{j}))[S_{\mathbf{i}+\mathbf{j}}, S_{\mathbf{k}}] \\
&= 2 \sin(\alpha_2(\mathbf{i},\mathbf{j}))[iS_{\mathbf{i}+\mathbf{j}}, S_{\mathbf{k}}] \\
&= 2 \sin(\alpha_2(\mathbf{i},\mathbf{j})) ad_{iS_{\mathbf{i}+\mathbf{j}}}(S_{\mathbf{k}}) \\
&= -2 \sin(\alpha_2(\mathbf{j},\mathbf{i})) e_{\mathbf{i}+\mathbf{j}}(S_{\mathbf{k}}) \qquad (2.112)
\end{aligned}
$$

As expected due to to the definition of the generators, the structure coefficients of the Lie algebra of derivations are

$$
\tilde{C}^{\mathbf{i}}_{\mathbf{mr}} = 2 \sin(\alpha_2(\mathbf{m},\mathbf{r}))\delta^{\mathbf{i}}_{\mathbf{m}+\mathbf{r}} = -iC^{\mathbf{i}}_{\mathbf{mr}}. \qquad (2.113)
$$

Since $\{dS_{\mathbf{i}}\}$ as a basis of the dual space is of minor importance, we define a 1-form basis $\{\theta^{\mathbf{i}}\}$ dual to the $e_{\mathbf{i}}$, as described in section 2.1:

$$
\theta^{\mathbf{i}} e_{\mathbf{j}} = \delta^{\mathbf{i}}_{\mathbf{j}} \mathbb{1} \qquad (2.114)
$$

which satisfies the exchange relation known from the commutative differential geometry

$$
\theta^{\mathbf{i}}\theta^{\mathbf{j}} = -\theta^{\mathbf{j}}\theta^{\mathbf{i}}. \qquad (2.115)
$$

Since we know the structure coefficients, we can directly write down the important quantities:

Equation (2.22) gives

$$
\begin{aligned}
d\theta^{\mathbf{i}} &= -\frac{1}{2}\tilde{C}^{\mathbf{i}}_{\mathbf{mr}}\theta^{\mathbf{m}}\theta^{\mathbf{r}} \\
&= -\sin(\alpha_2(\mathbf{m},\mathbf{r}))\delta^{\mathbf{i}-\mathbf{m}}_{\mathbf{r}}\theta^{\mathbf{m}}\theta^{\mathbf{r}} \\
&= -\sin(\alpha_2(\mathbf{m},\mathbf{i}))\theta^{\mathbf{m}}\theta^{\mathbf{i}-\mathbf{m}} \qquad (2.116)
\end{aligned}
$$

The canonical form is

$$
\theta = S_{\mathbf{i}}\theta^{\mathbf{i}} \qquad (2.117)
$$

The symplectic form is

$$\begin{aligned}\hat{\Omega} := d\theta &= \frac{1}{2}\tilde{C}^{\mathbf{i}}_{\mathbf{mr}} S_{\mathbf{i}} \theta^{\mathbf{m}} \theta^{\mathbf{r}} \\ &= \sum_{\mathbf{mri}} \sin\left(\alpha_2(\mathbf{m},\mathbf{r})\right) \delta^{\mathbf{i}-\mathbf{m}}_{\mathbf{r}} S_{\mathbf{i}} theta^{\mathbf{m}} \theta^{\mathbf{r}} \\ &= \sum_{\mathbf{mi}} \sin\left(\alpha_2(\mathbf{m},\mathbf{i})\right) S_{\mathbf{i}} \theta^{\mathbf{m}} \theta^{\mathbf{i}-\mathbf{m}} \\ &= \sum_{\mathbf{mi}} \sin\left(\alpha_2(\mathbf{m},\mathbf{i})\right) S_{\mathbf{m}+\mathbf{i}} \theta^{\mathbf{m}} \theta^{\mathbf{i}} \end{aligned} \tag{2.118}$$

according to equation (2.25).

If we compare this to the general expression for a 2-form

$$\hat{\Omega} = \sum_{\mathbf{mi}} \Omega_{\mathbf{mi}} \theta^{\mathbf{m}} \theta^{\mathbf{i}} \tag{2.119}$$

we can directly read the components of the associated hypermatrix $(\Omega_{\mathbf{kl}})$:

$$\Omega_{\mathbf{ij}} = 2\sin\left(\alpha_2(\mathbf{i},\mathbf{j})\right) S_{\mathbf{i}+\mathbf{j}} \tag{2.120}$$

It is the analogue to the symplectic matrix in classical mechanics. Before we examine the structure and meaning of Ω, let us see how its components can be used to describe quantities in our noncommutative differential geometry. Obviously, equation (2.120) equals

$$\Omega_{\mathbf{ij}} = i[S_{\mathbf{i}}, S_{\mathbf{j}}] = \tilde{C}^{\mathbf{k}}_{\mathbf{ij}} S_{\mathbf{k}}. \tag{2.121}$$

Since the elements are closely related to the commutator it is easy to write down some elemental expressions using them:

$$\begin{aligned}[\hat{A}, \hat{B}] &= N^{-2} A^{\mathbf{i}} B^{\mathbf{j}} [S_{\mathbf{i}}, S_{\mathbf{j}}] \\ &= \frac{1}{iN^2} A^{\mathbf{i}} B^{\mathbf{j}} \Omega_{\mathbf{ij}} \end{aligned} \tag{2.122}$$

$$\begin{aligned} e_{\mathbf{i}}(\hat{A}) &= i[S_{\mathbf{i}}, \hat{A}] \\ &= N^{-1} A^{\mathbf{j}} i [S_{\mathbf{i}}, S_{\mathbf{j}}] \end{aligned}$$

$$= N^{-1} \sum_{\mathbf{j}} A^{\mathbf{j}} \Omega_{\mathbf{ij}} \tag{2.123}$$

$$d\hat{A} = N^{-1} \sum_{\mathbf{mn}} A^{\mathbf{n}} \Omega_{\mathbf{mn}} \theta^{\mathbf{m}} \tag{2.124}$$

since $d\hat{A}(e_{\mathbf{m}}) = i[S_{\mathbf{m}}, \hat{A}]$.

Obviously for a field $X = N^{-1} X^{\mathbf{m}} e_{\mathbf{m}}$,

$$i_X \theta = \theta(X) = N^{-1} \sum_{\mathbf{m}} X^{\mathbf{m}} S_m. \tag{2.125}$$

Thus

$$\begin{aligned} d i_X \theta &= d(\theta(X)) \\ &= N^{-1} \sum_{\mathbf{m}} X^{\mathbf{m}} dS_m \\ &= N^{-2} \sum_{\mathbf{mk}} \Omega_{\mathbf{km}} X^{\mathbf{m}} \theta^{\mathbf{k}} \end{aligned} \tag{2.126}$$

using the special case of equation (2.124) where $A = S_{\mathbf{m}}$ Furthermore,

$$\begin{aligned} i_X \Omega &= N^{-1} \sum_{\mathbf{mk}} \Omega_{\mathbf{mk}} \theta^{\mathbf{m}}(X) \theta^{\mathbf{k}} \\ &= N^{-2} \sum_{\mathbf{mk}} \Omega_{\mathbf{mk}} X^{\mathbf{m}} \theta^{\mathbf{k}} \\ &= -N^{-2} \sum_{\mathbf{mk}} \Omega_{\mathbf{km}} X^{\mathbf{m}} \theta^{\mathbf{k}} \end{aligned} \tag{2.127}$$

It follows that for an arbitrary vector field X,

$$L_X \theta = 0 \text{ and} \tag{2.128}$$

$$\begin{aligned} L_X \Omega &= L_X(d\theta) \\ &= d(L_X \theta) \\ &= 0 \end{aligned} \tag{2.129}$$

hold. $i_X \Omega$ is always exact. The "generating function" is just an operator of the form

$\theta(X)$:

$$d(\theta(X)) = \Omega(X, \cdot) \tag{2.130}$$

Hence, if we view the quantum phase space in classical terms, every vector field is strictly Hamiltonian (see [AP95]) and all transformations in the QPS are canonical.

Let us calculate the inverse matrix $\hat{r}$ of $\hat{\Omega}$. The elements of the product hypermatrix should be $\delta^{\mathbf{i}}_{\mathbf{j}}\mathbb{1}$ - a hypermatrix with $\mathbb{1}$-matrices on its diagonal and $\hat{0}$ else.

$$r^{\mathbf{ij}}\Omega_{\mathbf{jk}} = \Omega_{\mathbf{ij}}r^{\mathbf{jk}} = \delta^{\mathbf{i}}_{\mathbf{k}}\mathbb{1} \tag{2.131}$$

Using equation (2.120) we get

$$\begin{aligned}
\Omega_{\mathbf{ij}}r^{\mathbf{jk}} &= \sum_{\mathbf{j}} 2\sin\left(\alpha_2(\mathbf{i},\mathbf{j})\right)S_{\mathbf{i}+\mathbf{j}}r^{\mathbf{jk}} \\
&= -i\sum_{\mathbf{j}}\left(e^{i\alpha_2(\mathbf{i},\mathbf{j})} - e^{-i\alpha_2(\mathbf{i},\mathbf{j})}\right)e^{i\alpha_2(\mathbf{i},\mathbf{j})}S_{\mathbf{i}}S_{\mathbf{j}}r^{\mathbf{jk}} \\
&= -i\sum_{\mathbf{j}}\left(e^{2i\alpha_2(\mathbf{i},\mathbf{j})} - 1\right)S_{\mathbf{i}}S_{\mathbf{j}}r^{\mathbf{jk}}
\end{aligned} \tag{2.132}$$

To get rid of the Schwinger elements, we define $r^{\mathbf{jk}} = cS^{\mathbf{k}}S^{\mathbf{j}}$. Remember, that $S^{\mathbf{j}} = S_{-\mathbf{j}}$. Thus

$$\begin{aligned}
\Omega_{\mathbf{ij}}r^{\mathbf{jk}} &= -i\sum_{\mathbf{j}}\left(e^{2i\alpha_2(\mathbf{i},\mathbf{j})} - 1\right)S_{\mathbf{i}}S_{\mathbf{j}}r^{\mathbf{jk}} \\
&= -ic\sum_{\mathbf{j}}\left(e^{2i\alpha_2(\mathbf{i},\mathbf{j})} - 1\right)S_{\mathbf{i}}S_{\mathbf{j}}S^{\mathbf{k}}S^{\mathbf{j}} \\
&= -ic\sum_{\mathbf{j}}\left(e^{2i\alpha_2(\mathbf{i},\mathbf{j})} - 1\right)S_{\mathbf{i}}S_{\mathbf{j}}e^{2i\alpha_2(\mathbf{j},\mathbf{k})}S^{\mathbf{j}}S^{\mathbf{k}} \\
&= -icS_{\mathbf{i}}S^{\mathbf{k}}\sum_{\mathbf{j}}\left(e^{2i\alpha_2(\mathbf{j},\mathbf{k}-\mathbf{i})} - e^{2i\alpha_2(\mathbf{j},\mathbf{k})}\right) \\
&= -icS_{\mathbf{i}}S^{\mathbf{k}}\sum_{\mathbf{j}}\left(e^{2i\alpha_2(\mathbf{j},\mathbf{k}-\mathbf{i})}\right) - \sum_{\mathbf{j}}\left(e^{2i\alpha_2(\mathbf{j},\mathbf{k})}\right)
\end{aligned} \tag{2.133}$$

Since index $\mathbf{0}$ is excluded, these are not quite the Kronecker delta expressions times N^2, but the missing constant addends would annihilate each other in the difference. The second factor is $\delta^{\mathbf{0}}_{\mathbf{k}}$ and thus can never be 1. So we can simply the whole expression to

$$\Omega_{\mathbf{ij}} r^{\mathbf{jk}} = -iN^2 c S_{\mathbf{i}} S^{\mathbf{k}} \delta^{\mathbf{i}}_{\mathbf{k}} \tag{2.134}$$

The ansatz for $r^{\mathbf{jk}}$ was correct and we can identify the constant c as i/N^2. The elements of the inverse hypermatrix are

$$r^{\mathbf{jk}} = \frac{i}{N^2} S^{\mathbf{k}} S^{\mathbf{j}} \tag{2.135}$$

Since we ignored the $\{dS_{\mathbf{i}}\}$ 1-form basis, these components do not play an important role in expressions of the NDG but do regarding the analogue to the classical theory. Indeed, there is an important difference: The inverse of the classical symplectic matrix is antisymmetric itself. In our noncommutative case, this is obviously not the case:

$$r^{\mathbf{jk}} = \frac{i}{N^2} S^{\mathbf{k}} S^{\mathbf{j}} = \frac{i}{N^2} e^{2i\alpha_2(\mathbf{j},\mathbf{k})} S^{\mathbf{j}} S^{\mathbf{k}} = e^{-2i\alpha_2(\mathbf{k},\mathbf{j})} r^{\mathbf{kj}} \tag{2.136}$$

The reason can be found in equation (2.133) were the noncommutative elements are chosen to annihilate the basis elements. If the basis did commute, the inverse of the symplectic matrix would be antisymmetric.

Let us have a final survey. Our phase space $\mathbb{Z}_n \times \mathbb{Z}_n$ has a group structure and its semiclassical continuum limit is Euclidean. As a consequence, all fields are strictly Hamiltonian. In contrast to the classical case, they are globally related to a generating "function" in form of an operator of the form $\theta(\hat{A})$.

To achieve the factor $i/\hbar$, we have to scale the canonical form with $\hbar$ and define the generators of $Der(M_N(\mathbb{C}))$ as $e_{\mathbf{m}} = ad(iS_{\mathbf{m}})$. Then, the Poisson bracket equivalent is defined as

$$\begin{aligned}
\{\hat{A}, \hat{B}\} &= \frac{i}{\hbar}[\hat{A}, \hat{B}] \\
&= \hbar\Omega(X_{\hat{A}}, X_{\hat{B}}) \\
&= \frac{2}{\hbar N^2} \sum_{\mathbf{m},\mathbf{n}} A^{\mathbf{m}} B^{\mathbf{n}-\mathbf{m}} sin[\alpha_2(\mathbf{m}, \mathbf{n})] S_{\mathbf{n}}
\end{aligned} \tag{2.137}$$

where $\hat{A} = N^{-1} \sum_{\mathbf{m}} A^{\mathbf{m}} S_{\mathbf{m}}$ ($\hat{B}$ likewise) and $X_{\hat{A}}$ ($X_{\hat{B}}$) is the corresponding Hamiltonian vector field

$$X_{\hat{A}} = -\frac{i}{N} \sum_{\mathbf{m}} A^{\mathbf{m}} e_{\mathbf{m}} \tag{2.138}$$

Due to the Lie group structure of the phase space, the Poisson bracket of operators even generates the commutator of the corresponding fields

$$[X_{\hat{A}}, X_{\hat{B}}](\cdot) = d\{\hat{A}, \hat{B}\}ad(i\cdot) \tag{2.139}$$

since

$$[e_{\mathbf{k}}, e_{\mathbf{l}}](S_{\mathbf{m}}) = (d\{S_{\mathbf{k}}, S_{\mathbf{l}}\})(e_{\mathbf{m}}) = e_{\mathbf{m}}(\{S_{\mathbf{k}}, S_{\mathbf{l}}\}) \tag{2.140}$$

2.3.2. Some words on the continuum limit

To bring this chapter to a close, we will briefly discuss how the the continuum limit is constructed. The method for the phase space is well known, e.g. from Fourier analysis.

With $N \to \infty$, every number of the form $\sqrt{2\pi/N}m$ becomes a real number a. Sums become integrals:

$$\sqrt{2\pi/N}\sum_{m=0}^{N-1} \to \int_{\mathbb{R}} da$$

Since the distance of two nearby lattice points is defined as $\sqrt{2\pi/N}$, the area of a phase space cell $\sqrt{2\pi/N} \to 0$ for $N \to 0$. “Nearby” in this context means elements which are *one step* in *one direction* apart. For an element (m, n) these are the four elements $(m+1, n)$, $(m-1, n)$, $(m, n+1)$ and $(m, n-1)$.

Thus, as mentioned before, the QPS is just the Euclidean plane in continuum limit.

The common choice (see [Fad95]) for the the operators $\mathcal{U}$ and $\mathcal{V}$ is

$$\mathcal{U} = e^{i\sqrt{2\pi/N}\hat{q}} \qquad \text{and} \tag{2.141}$$

$$\mathcal{V} = e^{i\sqrt{2\pi/N}\hat{p}} \tag{2.142}$$

Hence, the Schwinger elements are

$$\begin{aligned} S_{\mathbf{m}} &= \left(e^{2\pi i/N}\right)^{\frac{m_1 m_2}{2}} e^{i\sqrt{2\pi/N}m_1\hat{q}} e^{i\sqrt{2\pi/N}m_2\hat{p}} \\ &= e^{i\frac{\sqrt{2\pi/N}m_1\sqrt{2\pi/N}m_2}{2}} e^{i\sqrt{2\pi/N}m_1\hat{q}} e^{i\sqrt{2\pi/N}m_2\hat{p}} \end{aligned} \tag{2.143}$$

and in continuum limit

$$S_{\mathbf{a}} = e^{i\frac{a_1 a_2}{2}} e^{ia_1\hat{q}} e^{ia_2\hat{p}} \tag{2.144}$$

Let us examine α_2. In the limit it becomes

$$\alpha_2(\mathbf{a}, \mathbf{b}) = \frac{a_1 b_2 - a_2 b_1}{2} = \frac{1}{2}\det((\mathbf{ab})) \tag{2.145}$$

Since the determinant measures the (hyper) space spanned by the vectors forming the matrix, the 2-form α_2 gives (half) the area of the QPS enclosed by the vectors defining the related operators.

In the previous parts we found that the sign of some expressions is effected by the dimension of the lattice. Especially, the determinants of the Weyl operators differ between even and odd dimensions. Faddeev pointed out that this leads to a division of the degrees of freedom in continuum. While it is not relevant in one particle quantum mechanics, it is important in QFT (see [Fad95]).

We will stop our examination of the quantum phase space and its relations to classical mechanics at this point and make another approach in the next chapter.

3. The phase space in terms of fiber bundles

3.1. The idea

In the previous part we regarded the quantum phase space based on a noncommutative matrix differential geometry. With the Schwinger basis we examined a basis for the vector space of matrices. Multiplication of Schwinger elements leads to a phase factor and the addition of the indices.

These index pairs are located in the lattice torus $\mathbb{Z}_N \times Z_N$, which we identified with the quantum phase space. Since multiplication of elements leads to an addition of indices, it is not far to view these Schwinger elements to be part of a space attached to a position in the lattice and the multiplication as a transport action between these spaces.

Indeed, spaces "attached" to a point of another space are not an unknown field in mathematics. The corresponding structure is called a fiber bundle. Differential structures and transports have been extensively examined on bundles build of differentiable structures. However, in our case at least one structure – the phase space that constitutes the base space of the bundle – is not equipped with a differentiable structure but is discrete. Before we develop an analogon to the smooth version of transport, a short introduction to the common theory will be given.

3.2. An introduction to the theory

Before we start with the theory of discrete bundles a short introduction to the common theory of fiber bundles and connections on principal bundles will be given. References to the following content are [Ste74], [CBDM82] and [Nak90], the first one regarding the general topological structure, the latter ones for differential properties.

3.2.1. Fiber bundles

General definition

We start with the general definition of a fiber bundle.

Definition 4 *A (differentiable)* ***fiber bundle*** *(E, π, B, F, G) (or E or $E \xrightarrow{\pi} B$ in shorthand notation) consists of*

1. *a topological space (manifold) E, called the* ***total space****,*
2. *a topological space (manifold) B, called the* ***base space****,*
3. *a topological space (manifold) F, called the* ***fiber****,*
4. *a surjection π called* ***projection****, which maps E onto B in such a way, that $\pi^{-1}(p) \equiv F \cong F_p$ is the fiber in $p \in B$,*
5. *a family of open sets U_j covering B, called coordinate neighborhoods, with associated homeomorphisms (diffeomorphisms) $\phi_j : U_j \times F \to \pi^{-1}(U_j)$, called coordinate functions or local trivialisations, and*
6. *a topological group (Lie group) G, called the* ***structure group****, acting on F from the left.*

The coordinate functions are required to satisfy

$$\pi\phi_j(b, f) = b \tag{3.1}$$

and for $b \in U_i \cap U_j$

$$\phi_{j,b}^{-1}\phi_{i,b} := [\phi_j^{-1}(\phi_i(b,\Box))]_F : F \to F_b \tag{3.2}$$

– where $[\ ]_F$ means taking only the F component of the element in $U_j \times F$ – coincides with the left-action of an element $g_{ji}(b)$ of G on F. We will call these $g_{ij} : U_i \cap U_j \to G$ ***transition functions****.*

Mathematically strictly speaking, this is the definition of a coordinate bundle. Fiber bundles are defined independent of a special covering, by defining them as the equivalence class of coordinate bundles with identical (E, π, B, F, G), which form another coordinate bundle when uniting their sets of neighborhoods and local trivialisations (see [Nak90]):

$$\begin{aligned} &(E, \pi, B, F, G, \{U_i\}, \{\phi_i\}) \sim (E, \pi, B, F, G, \{V_i\}, \{\psi_i\}) \\ &\quad \Leftrightarrow (E, \pi, B, F, G, \{U_i\} \cup \{V_i\}, \{\phi_i\} \cup \{\psi_i\}) \text{ is a coordinate bundle} \end{aligned} \tag{3.3}$$

Since in physics, we always consider a special covering, we will also use the term fiber bundle for coordinate bundles in the following.

Obviously, if $E \cong B \times F$, it is possible to define a global coordinate function (the identity on $B \times F$). Then the structure group consists of the group identity alone.

We can visualize the fiber bundle as fibers attached to every point of the base space. A common and well-known example of fiber bundles - to be more precise, vector bundles as their fibers are vector spaces - is the **tangent bundle** TM of a manifold M, with M as base space and T_pM as fibers. The structure group is the full linear group acting on F, since the construction of the correspondences between the fibers is not unique.

Concerning the structure group, another example is more interesting. Consider S_1 both as base and as fiber space. We can choose G only to contain the identity. We then have just the plain product space $S_1 \times S_1$ which is easy to visualize. It is just the common torus. But this is by far not the only possibility:

We can build a Klein bottle by gluing the ends of the tube together in the other orientation. In this case, the structure group contains the identity and the negation of the coordinate of the circle, thus $G \cong \mathbb{Z}_2$.

This structure is topologically different from the torus, but that is not true for any two bundles with the same base, and fiber space, and different structure groups. If we apply the group containing the identity and the rotation by 180°, the torus will be twisted, but it is homeomorphic to the common torus.

Principal bundles

The special case of a fiber bundle which we will work on later is the principal bundle:

Definition 5 *A **principal bundle** is a fiber bundle with a structure group identical to the fiber. We will just write P(B,G) and call it a **G bundle**.*

In addition to the left action of the group on the fiber, we can now define an additional right action on the bundle by

$$R_h e = eh = \phi_i(b, gh) \tag{3.4}$$

for $\phi_i^{-1}(e) = (b, g)$, $e \in E, g, h \in G = F$. So, the right action of h on the bundle means the right action on the fiber space in in the locally homeomorphic Cartesian product.

Let us consider two local trivialisations ϕ_i, ϕ_j on overlapping neighborhoods U_i and U_j, $U_i \cap U_j \neq \varnothing$. Then

$$eh = \phi_i(b, g_i h) = \phi_j(b, g_{ji}(b)(g_i h)) = \phi_j(b, (g_{ji}(b) g_i) h) = \phi_j(b, g_j h). \tag{3.5}$$

So, since the left action of the transition functions commutes with the right action of h, the latter is independent of the local trivialisations.

Cross sections

We will now consider maps associated to fiber bundles. A special case is the section or cross section:

Definition 6 *A **cross section** σ on a fiber bundle $E \xrightarrow{\pi} B$ is a continuous map $\sigma : B \longmapsto E$ satisfying $\pi \circ \sigma = id_B$.*

It can be regarded as a generalized graph of a function. Cross section always refers to a globally defined map in this work. Sections only defined on neighborhoods are called **local sections** here.

Figure 3.1 shows an example of a section on the Möbius strip, described as a non-trivial fiber bundle $MS \xrightarrow{\pi} S_2$ with fibers $[-1, 1]$.

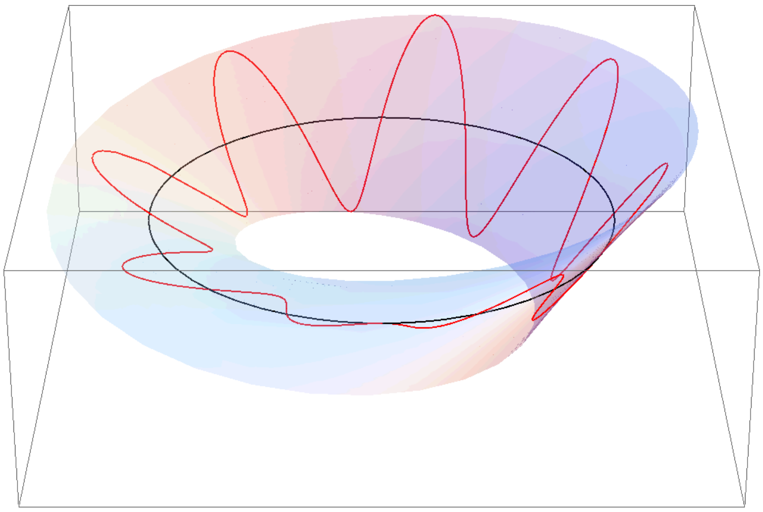

Figure 3.1.: Section on Möbius strip. Base manifold black, section red.

In the following, we will refer to the set of all sections on M (mapping to E) as $\Gamma(M, E)$. In the special case that $E = TM$, $\Gamma(M, TM)$ is obviously identified with the set of vector fields $\mathcal{X}(M, TM)$.

Cross sections may not exist on a base space in general. Indeed, the following theorem applies:

Theorem 1 *A principal bundle is trivial, if and only if it admits a global section.*

For a *proof*, which will not take us any further here, see [Nak90, section 9.4.2]. Note however that this is by no means true for general fiber bundles. In figure 3.1 we have already shown a global section on a nontrivial bundle. Another example is the null section $\sigma_0 \in \mathcal{X}(M, TM)$, for which $\phi_i^{-1}(\sigma_0(x)) = (x, 0)$ holds and which can be defined on every vector bundle.

3.2.2. Connections on fiber bundles

Definitions of a connection on a principal fiber bundle

As we need a rather graphic picture of connections for our reduction to discrete base space bundles in the further details, we will invoke the geometrical definition of connections on fiber bundles as for example presented in [Nak90] and [CBDM82]. For the classical connections we will need a differentiable principal bundle $P := P(M, G)$, see definition given above.

We separate the tangent space T_pP of P in a point p (not to be confused with the tangential space T_xM of the base manifold M) into a horizontal and a vertical subspace H_pP and V_pP.

The definition of vertical subspace is inherent in the structure of the bundle itself. It can be visualized as the part of T_pP tangent to the fibers G_x with $x = \pi(p)$, so it can be identified with T_gG with $g \in G_x$.

We will show that $V_pP \cong \mathcal{L}(G)$, the Lie algebra of G. Consider a mapping

$$\mathbb{R} \times \mathcal{L}(G) \longrightarrow G\colon (t, A) \longmapsto \exp tA. \tag{3.6}$$

It defines an Abelian one-parameter subgroup of G for a given A - i.e. a curve in G parametrized by t - or a one-to-one mapping from $\mathcal{L}(G)$ to G for a given t (which is a well-known theorem from the theory of Lie groups and Lie algebras, see e.g. [Nak90, pp. 173ff]). Using this, the definition

$$A^{\#} f(p) = \frac{d}{dt} f(p \exp(tA))|_{t=0}, f \in \mathcal{F}(P) \tag{3.7}$$

provides a one-to-one mapping from $\mathcal{L}(G)$ to T_pP by defining $A^{\#} \in T_pP$ as the derivative along the curve $\phi_A(t) = p \exp(tA)$ at $t = 0$ (obviously, $\phi_A(0) = p$). $\phi_A(t)$ obviously lies completely in G, since the action of $exp(tA) \in G$ preserves the fiber.

Thus $A^{\#} \in V_pP$. We construct a vector field $A^{\#}$ over P by applying this definition to every point in P. It is called the **fundamental vector field**.

$V_pP \cong T_gG \cong T_eG$ (where $x = \pi(p)$, $g \in G_x$ and e is the identity of G) and L_g defines a canonical isomorphism between T_eG and $\mathcal{L}(G)$. It follows directly that $\dim(\mathcal{L}(G)) = \dim(V_pP)$. Hence – as it is also one-to-one – $\# : A \longmapsto A^\#$ is an isomorphism. We will call the inverse mapping $\#^{-1} : A^\# \longmapsto A^{\#\#^{-1}} = A$.

It is easy to see that, as for all vectors in V_pP, $\pi_* A^\# = 0$ (where π_* is the differential map induced by π): Just apply the definition

$$\pi_* A^\#[f](u) = \frac{d}{dt} f(\pi(u \exp tA))|_{t=0} = \frac{d}{dt} f(\pi(u))|_{t=0} = 0 \tag{3.8}$$

(where now $f \in \mathcal{F}(M)$) since the right action does not change the fiber.

The horizontal subspace of T_pP is not uniquely defined by the bundle structure. The separation of the tangential space is done by defining a **connection** on $P(M,G)$. There are different ways to do that. Let us first have a look at the way which directly defines a connection by the choice of the horizontal space:

Definition 7 *A **connection** an a principal bundle $P(M,G)$ is a unique separation of T_pP into fields of vector spaces H_pP and V_pP, $H_pP, V_pP \subset T_pP$, such that*

1. *H_pP depends differentiably on p,*
2. *$T_pP = H_pP \oplus V_pP$,*
3. *every smooth vector field $X \in TP$ can be separated into a smooth horizontal vector field $X_H \in HP$, the bundle of horizontal subspaces on P, and a smooth vertical vector field $X_V \in VP$, the bundle of vertical subspaces on P, with $X_H + X_V = X$ and*
4. *a horizontal subspace H_pP in $p \in P$ is related to all other horizontal subspaces on the same fiber by the right action of the group:*

$$R_{g*} H_pP = H_{R_g p}P \tag{3.9}$$

This is a formulation quite easy to visualize. But as we are heading towards curvature in terms of differential forms, it is convenient to choose another definition.

As outlined before, we have an isomorphism between V_pP and $\mathcal{L}(G)$ by $A^\# \mapsto A$. This induces a linear mapping $T_pP \to \mathcal{L}(G)$ by simply first taking the vertical component of a vector before applying $\#^{-1}$. On the full tangential bundle we call this mapping a 1-form ω on P with values in $\mathcal{L}(G)$ (which means it is in $\mathcal{L}(G) \otimes \Omega^1 P$). Given a basis $\{e_\alpha\}$ of $\mathcal{L}(G)$ and a basis $\{\theta^b\}$ of $\Omega^1 P (= T^*P)$, we can write ω as

$$\omega = \omega_b^\alpha e_\alpha \otimes \theta^b \tag{3.10}$$

Because of (1) in definition (7), the ω_b^α are differentiable functions on P.

By definition, it is obvious that

$$\omega(X_H + X_V) = \omega(X_V) \tag{3.11}$$

and that

$$\omega(A^\#) = A \tag{3.12}$$

.

Since $\mathcal{L}(G) \cong T_e G$ and after restriction to $V_p P \cong T_g G$, we can interpret ω as the pushforward of a vector in G along the left-translation in the group into $T_e G$:

$$\omega(X_g) = L_{g^{-1}*} X_g, \tag{3.13}$$

where $X_g \in T_g G_{\pi(p)}$ and thus $L_{g^{-1}*} X_g \in T_e G_{\pi(p)}$.

Such a form is well-known as the canonical (or Maurer-Cartan) form on G. It carries the basic infinitesimal information about the structure of G, and follows the transformation law

$$R_g^* \omega = Ad_{g^{-1}} \omega, \tag{3.14}$$

as we can easily check:

$$\begin{aligned} R_h^* \omega(X_{gh^{-1}}) &= \omega(R_{h*} X_{gh^{-1}}) \\ &= L_{g*}^{-1} R_{h*} X_{gh^{-1}} \\ &= L_{h*}^{-1} L_{gh^{-1}*}^{-1} R_{h*} X_{gh^{-1}} \\ &= L_{h*}^{-1} R_{h*} L_{gh^{-1}*}^{-1} X_{gh^{-1}} \\ &= L_{h^{-1}*} R_{h*} \omega(X_{gh^{-1}}) \\ &= Ad_{g^{-1}} \omega(X_{gh^{-1}}) \end{aligned} \tag{3.15}$$

Since the transformation law above is also trivially fulfilled by horizontal vectors, we come to the following new definition:

Definition 8 *A connection on a principal bundle $P(M, G)$ is a $\mathcal{L}(G)$-valued 1-form $\omega \in \mathcal{L}(G) \otimes \Omega^1 P$ obeying the following properties:*

1. *$\boldsymbol{\omega}_p \in T_p^* P$ depends differentiably on p.*
2. *$\boldsymbol{\omega}$ projects the fundamental vector field $A^\#$ on its generating left-invariant vector field A: $\boldsymbol{\omega}(A^\#) = A$.*
3. *The pullback of the right action on the bundle R_g^* corresponds to the differential map of the adjoint action $Ad_{g^{-1}}$: $R_g^* \boldsymbol{\omega} = Ad_{g^{-1}} \boldsymbol{\omega}$.*

The horizontal space is then defined as the kernel of $\boldsymbol{\omega}$.

The way back to the first definition is easily shown by calculation:

Define $H_p P$ as the kernel of $\boldsymbol{\omega}$ and take $X \in H_p P$. Then

$$\boldsymbol{\omega}(R_{g*} X) = R_g^* \boldsymbol{\omega}(X) \overset{Def.(8).3}{=} g^{-1} \boldsymbol{\omega}(X) g = g^{-1} 0 g = 0. \tag{3.16}$$

So, since R_g is invertible, it follows that every $Y \in H_{ph} P$ is expressed through $R_h Z$ for some $Z \in H_p P$. Def. (7).4 follows directly.

As the remaining requirements have already become clear in the previous, the equivalence between the two definitions is shown.

As hinted before, this connection is by no means unique in general. Indeed, as proven for example in [CBDM82], on a principal bundle with paracompact base-space, an infinite number of connections.

Parallel transport

After having defined the horizontal subspace of $T_p P$, we can proceed to examine parallel transport of points $p \in P(M, G)$ between fibers.

Definition 9 *Consider a curve $C : [0, 1] \longrightarrow M$. We call a curve $\tilde{C} : [0, 1] \longrightarrow P$ with $\pi \circ \tilde{C} = C$ a **lift** of C.*

Definition 10 *We call a lift of C a* ***horizontal lift****, if in addition for all $t \in [0,1]$ applies: $T_{C(t)}C([0,1]) \subset H_{C(t)}P$, i.e. all the tangential vectors of $C(t)$ lie in the horizontal space.*

For the definition of parallel transport as transport along this curve to make sense, we have to prove its uniqueness:

Theorem 2 *Let $C : [0,1] \longrightarrow M$ be a curve with starting point x_0 and $p_0 \in P$ with $x_0 = \pi(p_0)$ a point. Then there is one and only one horizontal lift $\tilde{C} : [0,1] \longrightarrow P$ starting in p_0 for a given connection.*

The proof is easy. Since every tangential vector X lies in the horizontal subspace, $\omega(X) = 0$ in every point, which is a first order ordinary differential equation with a single boundary condition. The existence and uniqueness of such an equation is guaranteed by the theory of differential equations. □

Time for an example: Let us take the $\mathbb{R}$-bundle $P(\mathbb{R}, \mathbb{R}) \cong \mathbb{R} \times \mathbb{R}$ with the "fiber-$\mathbb{R}$" considered as an additive group. This bundle is obviously trivial and we choose *id* as local trivialization: $\phi : (x, f) \mapsto (x, f)$.

Let us define a first connection by

$$\omega_1 = df \tag{3.17}$$

Obviously for $A^{\#} = A\partial/\partial f$, where A is in the Lie algebra $\mathcal{L}(\mathbb{R}) \cong \mathbb{R}$ of the additive group $\mathbb{R}$, $\omega_1(A^{\#}) = A$. Also, since $\mathbb{R}$ is abelian, $R_{g*}\omega_1 = g^{-1}\omega_1 g = \omega_1$. We can now calculate a horizontal lift $\tilde{C}_1$ of the curve $C \equiv id|_{[-1,1]}$, for example starting at $(-1, -1)$. Let X be tangent to $\tilde{C}_1$:

$$X = \frac{d}{dt} = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{df}{dt}\frac{\partial}{\partial f}. \tag{3.18}$$

Then

$$0 \stackrel{!}{=} \omega_1(X) = \frac{df}{dt}, \tag{3.19}$$

so $f(t) = const$. With the boundary condition we get $\tilde{C}_1 : t \mapsto (t, -1)$. The curve is shown blue in figure (3.2).

But as said, this is not the only possible connection. Let us define another one:

$$\omega_2 = df - dx, \tag{3.20}$$

which leads us to the differential equation

$$0 \overset{!}{=} \omega_2(X) = \frac{df}{dt} - \frac{dx}{dt} = \frac{df}{dt} - 1 \tag{3.21}$$

and the resulting horizontal lift $\tilde{C}_1 : t \mapsto (t, t)$ shown red in figure (3.2).

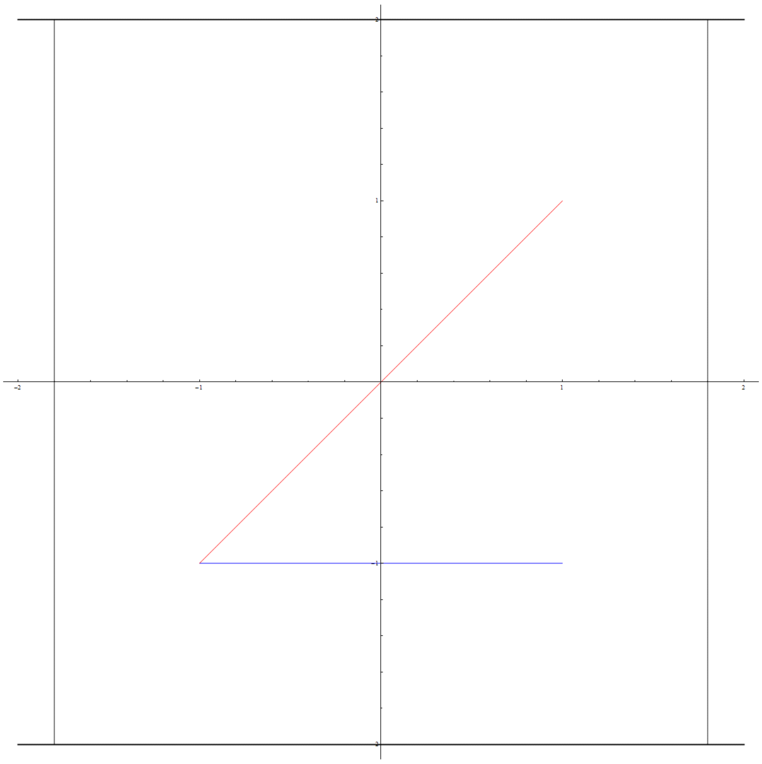

Figure 3.2.: Horizontal lifts on $P(\mathbb{R}, \mathbb{R})$.

We can now define a parallel transport along a horizontal lift:

Definition 11 *A point p_1 is called a* ***parallel transport*** *of p_0 along a curve $\tilde{C} : [0,1] \longrightarrow P$, if $\tilde{C}(t)$ is the horizontal lift a a curve $C : [0,1] \longrightarrow M$ through p_0 and $p_0 = \tilde{C}(0)$ and $p_1 = \tilde{C}(1)$.*

We can formulate this as a map $\Gamma[\tilde{C}] : \pi^{-1}(C(0)) \longrightarrow \pi^{-1}(C(1))$, which maps a point p_0 in $\pi^{-1}(C(0))$ to the point p_1 in $\pi^{-1}(C(0))$ on the unique horizontal lift $\tilde{C}$

through p_0. We use the lift instead of the curve as an argument since the transport map does not only depend on the curve but also on the connection.

It is obvious that

$$\Gamma[\tilde{C}]^{-1} = \Gamma[\tilde{C}^{-1}], \text{ where } \tilde{C}^{-1}(t) = \tilde{C}(1-t) \tag{3.22}$$

Let us study how horizontal lifts to the same curve C, but through different points p_0, p_0', are connected:

Let $\tilde{C}$, $\tilde{C}'$ be horizontal lifts of a curve $C \in M$ with $\tilde{C}(0) = \tilde{C}'(0)g$. Consider now the curve $t \mapsto \tilde{C}(t)g$. Since the horizontal space is right invariant according to definition 7 (4), the tangent vectors of this curve are also horizontal, thus it is a horizontal lift. So with theorem 2, which states the uniqueness of a horizontal lift through a given point, we find the following

Corollary 1 *Every point p on a horizontal lift $\tilde{C}$ of C is connected to the point p' on another horizontal lift $\tilde{C}'$ of C on the same fiber by the right action of the same group element g.*

This corollary ensures directly that your formerly defined parallel transport map $\Gamma[\tilde{C}]$ commutes with the right action on the bundle:

$$R_g \Gamma[\tilde{C}] = \Gamma[\tilde{C}] R_g \tag{3.23}$$

Holonomy

The parallel transport map $\Gamma[\tilde{C}]$ not only depends on the choice of the connection on the bundle, but also on the curve which is lifted. Consider two curves C, C'in M with $C(0) = C'(0)$ and $C(1) = C'(1)$. In general, $\Gamma[\tilde{C}](p_0)$ will not be equal to $\Gamma[\tilde{C}'](p_0)$. Thus, for a closed loop

$$L(t) = C * C'(t) = \begin{cases} C(2t) & t \in [0, 0.5) \\ C'(2(1-t)) = C'^{-1}(2(t-0.5)) & t \in [0.5, 1] \end{cases} \tag{3.24}$$

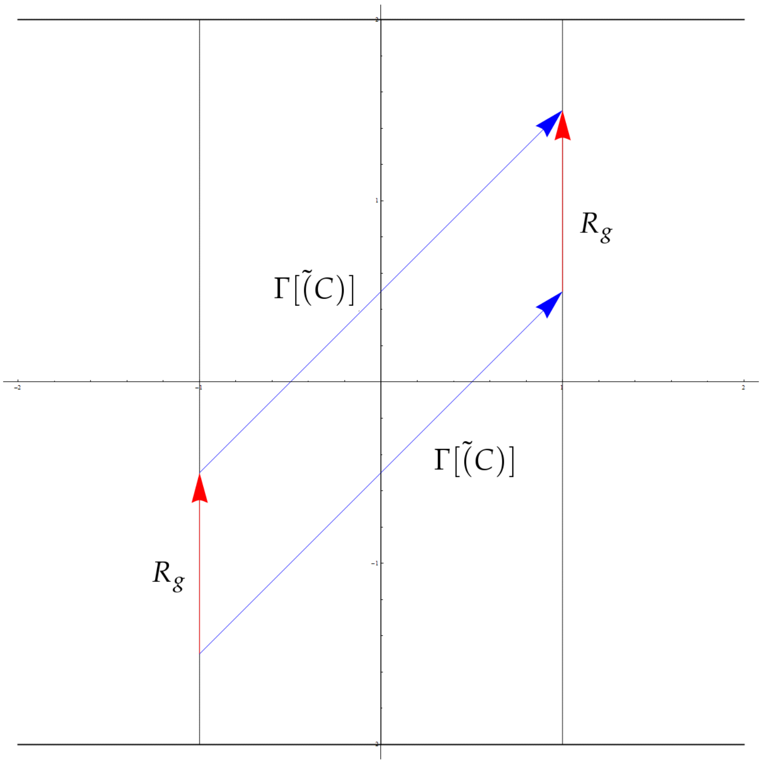

Figure 3.3.: Parallel transport commutes with right action.

(meaning $L(0) = L(1)$) $\tilde{L}(0) \neq \tilde{L}(1)$ and $\Gamma[\tilde{L}]$ defines a transformation $\tau[\tilde{L}]$ on $\pi^{-1}(L(0))$ which is not $id_{\pi^{-1}(L(0))}$ in general.

Because of equation (3.23) it is obvious that this transformation is compatible with the right action:

$$\tau[\tilde{L}](pg) = \tau[\tilde{L}](p)g \tag{3.25}$$

Since τ preserves the fiber, every transformation can be substituted by an element $g_{\tilde{L}}$ of the structure group:

$$\tau[\tilde{L}](p) = pg_{\tilde{L}} \tag{3.26}$$

Hence the set $L_\pi(p_0)M$ of all possible closed loops in M through $\pi(p_0)$ together with a connection defines a subset of the structure group G:

$$\Phi_{p_0} = \{g \in G | \tau[\tilde{L}](p_0) = p_0 g, \tilde{L} \in \tilde{L}_{p_0}M\} \tag{3.27}$$

where $\tilde{L}_{p_0}M$ is the set of horizontal lifts of the elements of $L_{\pi(p_0)}M$ through p_0.

It is easy to see that Φ_{p_0} is even a sub*group* of G:

- Φ_{p_0} contains the identity from the constant loop $L(t) = p_0$,
- it contains the inverse element of every $g_{\tilde{L}}$ as $g_{\tilde{L^{-1}}}$ and
- since $ug_{\tilde{L'}}g_{\tilde{L''}} = \tau[\tilde{L'}](p_0)g_{\tilde{L''}} = \tau[\tilde{L'}] \circ \tau[\tilde{L''}](u_0) = \tau[L' \tilde{*} L''](p_0)$ (where $L' * L''$ is the product of curves shown in (3.24)) and the product of loops is a loop, it is closed under the group operation.

We call it the **holonomy group** in p.

Curvature

In Riemannian geometry, it is well known that the curvature tensor is closely related to parallel transport of vectors along loops. In contradiction to the usual Euclidean geometry, on a curved manifold the resulting vector will in general be different from the original one.

In the following section, we will study curvature of principal bundles on an infinitesimal level at first and then reveal its relation to the formerly discussed holonomy through the Ambrose-Singer theorem.

Definition 12 *ϕ be a $\mathcal{L}(G)$-valued r-form on TP. We will call*

$$D\phi(v_1, ..., v_{r+1}) = d\phi(v_1^H, ..., v_{r+1}^H) \tag{3.28}$$

the ***exterior covariant derivative*** *of ϕ, where v_i^H means the horizontal component of v_i.*

Definition 13 *The exterior covariant derivative of the connection 1-form $\boldsymbol{\omega}$, $\boldsymbol{\Omega} = D\boldsymbol{\omega}$, is called the* ***curvature form*** *of the connection $\boldsymbol{\omega}$.*

Since the proof will provide us some hints for the geometrical meaning of $\boldsymbol{\Omega}$, we will consider the **Cartan structure equation** for a connection $\boldsymbol{\omega}$.

Let us first prove a short

Lemma 1 *Let $X \in H_pP$, $Y \in V_pP$. Then $[Y, X]$ is horizontal.*

Proof: Chose a curve $g(t)$ in P in such a way that it generates Y. Then by definition of $[\_,\_]$

$$[Y,X] = \lim_{t\to 0} t^{-1}(R_{g(t)*}X - X) \tag{3.29}$$

From the definition of the connection we know that the right action preserves the horizontal space, so $[Y,X] \in H_pP$. $\square$

Theorem 3 ***Cartan structure equation.*** *Let $X, Y \in T_pP$. If $\boldsymbol{\omega}$ is a connection on P, then*

$$\boldsymbol{\Omega} = D\boldsymbol{\omega} = d\boldsymbol{\omega}(X,Y) + [\boldsymbol{\omega}(X), \boldsymbol{\omega}(Y)]. \tag{3.30}$$

Let us consider $\boldsymbol{\Omega}(X,Y) = \boldsymbol{\Omega}(X^H + X^V, Y^H + Y^V)$, which – since $\boldsymbol{\Omega}$ is a skew-symmetric bilinear form – equals

$$\boldsymbol{\Omega}(X^H, Y^H) + \boldsymbol{\Omega}(X^H, Y^V) - \boldsymbol{\Omega}(Y^H, X^V) + \boldsymbol{\Omega}(X^V, Y^V).$$

Obviously, the last three addends are 0, since $(X^V)^H = (Y^V)^H = 0$. We will show that this also applies to the RHS of the structure equation.

Consider the case, in which the first argument is a horizontal and the second is a vertical vector. Let $X \in H_pP$, $Y \in V_pP$. Since $\boldsymbol{\omega}(X) = 0$ the commutator is 0 and it leaves us

$$0 \stackrel{!}{=} d\boldsymbol{\omega}(X,Y) = X\boldsymbol{\omega}(Y) - Y\boldsymbol{\omega}(X) - \boldsymbol{\omega}([X,Y]) = X\boldsymbol{\omega}(Y) - \boldsymbol{\omega}([X,Y]), \tag{3.31}$$

where we used the well-know property of the derivation of $p-1$-forms (see e.g. [CBDM82, p. 207])

$$\begin{aligned} d\boldsymbol{\omega}(V_0, V_1, ..., V_p) &= \sum_{i=0}^{p}(-1)^i V_i(\boldsymbol{\omega}(V_0, ..., V_{i-1}, V_{i+1}, ..., V_p)) \\ &+ \sum_{0\le i<j\le p}(-1)^{i+1} V_i(\boldsymbol{\omega}([V_i, V_j], V_0, ..., V_p)). \end{aligned} \tag{3.32}$$

Using lemma 1, $\boldsymbol{\omega}([X,Y])$ is 0. Since $\boldsymbol{\omega}(Y)$ is a constant element of $\mathcal{L}(G)$ it also equals 0.

Consider the case $X, Y \in V_pP$. Using the same arguments as above, it remains to show that

$$0 \stackrel{!}{=} -\boldsymbol{\omega}([X,Y]) + [\boldsymbol{\omega}(X), \boldsymbol{\omega}[Y]]. \tag{3.33}$$

Take $\boldsymbol{\omega}([X,Y]) = [X,Y]^{\#} \in \mathcal{L}(G)$ as defined above. Then, using the properties described in the former sections,

$$\boldsymbol{\omega}([X,Y]) = [X,Y]^{\#^{-1}} = [X^{\#^{-1}}, Y^{\#^{-1}}] = [\boldsymbol{\omega}(X), \boldsymbol{\omega}(Y)] \tag{3.34}$$

The last case to study is $X, Y \in H_pP$. There – since $\boldsymbol{\omega}(X) = \boldsymbol{\omega}(Y) = 0$ – we have

$$\boldsymbol{\Omega}(X,Y) = d\boldsymbol{\omega}(X,Y) = d\boldsymbol{\omega}(X^H, Y^H) \tag{3.35}$$

which is exactly the definition of the covariant exterior derivative stated before. □

We can also write the Cartan structure equation in terms of a basis $\{\boldsymbol{e}_\alpha\}$ of $\mathcal{L}(G)$.

Since

$$\begin{aligned}
[\boldsymbol{\omega}(X), \boldsymbol{\omega}(Y)] &= \frac{1}{2}([\boldsymbol{\omega}(X), \boldsymbol{\omega}(Y)] - [\boldsymbol{\omega}(Y), \boldsymbol{\omega}(X)]) \\
&= \frac{1}{2}[T_\beta, T_\gamma](\boldsymbol{\omega}^\beta(X)\boldsymbol{\omega}^\gamma(Y) - \boldsymbol{\omega}^\gamma(X)\boldsymbol{\omega}^\beta(Y)) \\
&= \frac{1}{2}[T_\beta, T_\gamma]\boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma(X,Y) \\
&= \frac{1}{2}c^\alpha_{\beta\gamma}\boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma \boldsymbol{e}_\alpha(X,Y)
\end{aligned} \tag{3.36}$$

we get

$$\boldsymbol{\Omega}^\alpha = d\boldsymbol{\omega}^\alpha + \frac{1}{2}c^\alpha_{\beta\gamma}\boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma. \tag{3.37}$$

Using this, we can show the following

Theorem 4 *$\boldsymbol{\omega}$ fulfills the **Bianchi identity** for the exterior covariant derivative*

$$DD\boldsymbol{\omega} = D\boldsymbol{\Omega} = 0 \tag{3.38}$$

Derivation of (3.37) gives

$$d\boldsymbol{\Omega}^\alpha = \frac{1}{2}c^\alpha_{\beta\gamma}d\boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma - \frac{1}{2}c^\alpha_{\beta\gamma}\boldsymbol{\omega}^\beta \wedge d\boldsymbol{\omega}^\gamma. \tag{3.39}$$

Since every summand contains $\boldsymbol{\omega}$ as a factor, which vanishes on horizontal vectors, the theorem follows immediately. □

Let us get back to our geometrical considerations. Take an infinitesimal parallelogram with corners $O = \{0, 0, 0, ..., 0\}$, $P = \{\epsilon, 0, 0, ..., 0\}$, $Q = \{\epsilon, \delta, 0, ..., 0\}$ and $R = \{0, \delta, 0, ..., 0\}$ in a coordinate system $\{x_i\}$ on a chart $U \in M$ with a set of coordinate basis vectors $\{\partial/\partial x_i\}$. It builds a loop $L \in M$. Choose $X, Y \in H_pP$ such that $\pi_* X = \epsilon\partial/\partial x_1$ and $\pi_* Y = \epsilon\partial/\partial x_2$. Obviously, X, Y are tangent to the horizontal lift $\tilde{L}$ of L.

We can investigate the direction of the $[X, Y]$:

$$\pi_*([X,Y]^H) = [\pi_* X^H, \pi_* Y^H] = [\pi_* X^H, \pi_* Y^H] = \epsilon\delta[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}] = 0 \tag{3.40}$$

Since π_* is an isomorphism between H_pP and T_pM, this means that $[X, Y]^H = 0$, so $[X, Y] \in V_pP$.

$\tilde{L}$ will not close in general. The vector $[X, Y]$ is (indeed proportionally) related to the distance between the initial and the final point on the same fiber. We can measure it using the connection 1-form $\boldsymbol{\omega}([X, Y])$. This, as we learned in the proof of the Cartan structure equation, is exactly the negative of the curvature 2-form for two horizontal vectors.

So, the element

$$A = -\boldsymbol{\omega}([X, Y]) = \boldsymbol{\Omega}(X, Y) \text{ with } X, Y \in H_pP \tag{3.41}$$

in $\mathcal{L}(G)$ is closely related to the non-closing of loops.

The exact relation is given by the the following

Theorem 5 *(**Ambrose-Singer**) Let $P(M, \pi, G)$ be a principal fiber bundle with connection $\boldsymbol{\omega}$ over a connected manifold M. The Lie algebra of an holonomy group Φ_{p_0} is equal to* *the subspace of $\mathcal{L}(G)$ spanned by all elements of the form*

$$\mathbf{\Omega}_p(X, Y) \text{ with } X, Y \in H_p P, \tag{3.42}$$

where p is an arbitrary point of P which can be obtained by parallel transport of p_0, i.e. $p \in P(p_0)$, the holonomy bundle at p_0.

A proof would go beyond the scope of this work. It is presented in [CBDM82].

3.3. The structure

3.3.1. Survey

Let us make a survey about what parts of the previously presented theory we have, and which we do not have.

On the one hand, we can easily define a full fiber bundle structure. We have

- $\mathbb{Z}_N \times \mathbb{Z}_N$ a *base space* B,
- a group G containing the Schwinger elements $\{S_{\mathbf{m}}\}$ (we will discuss this in detail later on) as a *fiber* B and
- the Cartesian product $B \times F$ as a *total space* E.
- The *projection* $\pi : E \to B$ just maps $(b, f) \in B \times F$ to $b \in B$.
- The bundle is trivial, hence it can be covered with a global *neighborhood* and the *coordinate functions* are just the identity functions.
- Since we want to achieve an analog with principal bundles, we choose G as the *structure group*, which is obviously compatible with the coordinate functions above.

Since the structure group is identical to the fiber, we can define a right and a left action of the group on the bundle space, which is well known from common principal bundles:

$$R_h e = eh = (b, gh) \tag{3.43}$$

$$L_h e = he = (b, hg) \tag{3.44}$$

for $e = (b, g) \in E$, $b \in B$ and $g, h \in G$.

Cross sections can be defined on the base space. Since B is discrete every map $B \to E$ is continuous and consequently a cross section.

On the other hand, we lack an important quantity for studying connections on the bundle. Since at least our base space is discrete it has no tangent space and consequently neither has the total space.

For G there are at least two groups to take into consideration:

1. The Heisenberg group introduced in subsection 2.2.3 and
2. the N-dimensional unitary group $U(N)$.

One is a Lie group and consequently has a tangent space, the other is discrete and has not. We will decide later which one to choose.

Thus, there is no common structure to divide into a horizontal and vertical part. We will have to define some new properties in our bundle.

3.3.2. New properties in the bundle

We will first examine to what extent we can copy a concept like the tangential space. An acceptable choice could be to consider the difference vectors between two nearby elements in the group divided by their distance in the lattice. This would provide a difference quotient that becomes the differential quotient of the derivation in the continuum limit. The choice of G would reasonably be the Heisenberg group to stay discrete in every part.

The difference quotients would then have a form like

$$\frac{((m,n),\omega^p S_{q,r}) - ((m,n),\omega^p S_{q-1,r})}{\sqrt{2\pi/N}} \text{ and} \tag{3.45}$$

$$\frac{((m,n),\omega^p S_{q+1,r}) - ((m,n),\omega^p S_{q,r})}{\sqrt{2\pi/N}} \tag{3.46}$$

Unfortunately, it is obvious that these vectors are not linearly dependent in the finite dimensional case. This is not suprising, as one can easily visualize by considering a function $\mathbb{R} \to \mathbb{R}$ (see figure 3.4).

Because of these problems, we abandon this approach and use another one in the following.

While sections can be defined in the well-known way, lifts cause problems, since on a discrete space, there are no curves. To solve this problem we invoke graph theory. We replace the curves by introducing undirected edges between nearby

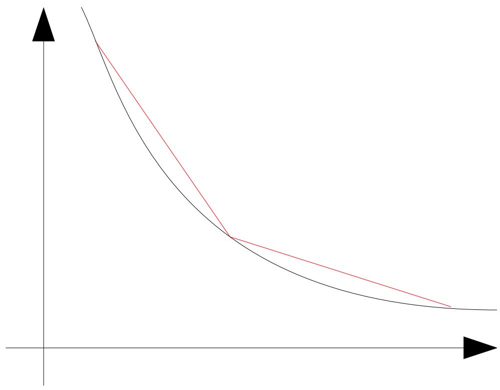

Figure 3.4.: Secants of a function graph are not linearly dependent in general. They are only tending towards linear dependence in their limit.

lattice points, making the the QPS an undirected graph. Nearby again means elements which are one step apart in one direction, thus for an element (m, n) these are $(m + 1, n)$, $(m - 1, n)$, $(m, n + 1)$ and $(m, n - 1)$. We get the following

Definition 14 *The analog of a curve in the continuous base space is call a* ***path*** *in the "base graph". We denote it by the map* $P : \{0, 1, \dots, n\} \ni m \mapsto l_m \in B$*, where* l_0 *is arbitrary and* l_{m+1} *is a nearby point of* l_m*, or just by the ordered set of points* $\langle l_0, \dots, l_n \rangle$*.*

In contrast to what paths usually refer to in graph theory, we do not demand these paths to be simple, thus they may contain a single vertex more than once. We will also omit the edges since they are of no further importance and are uniquely defined by the vertices in our special graph. If the path starts in the same vertex as it ends we will call it a cycle. (For an introduction in graph theory, see [Die05].)

We can now easily define a lift of a path:

Definition 15 *A map* $\tilde{P} : \{0, 1, \dots, n\} \ni m \mapsto p_m \in E$ *is called a* ***lift of the path*** P *if and only if* $\pi \circ \tilde{P} = P$*.*

Since we lack a tangent space we lack a connection in the classical sense. Thus we cannot apply the previous definition of a horizontal lift. We may proceed the other way round:

Instead of defining "horizontal" by the connection, we define the connection by a lift which we would like to call horizontal. Since a horizontal lift induces parallel

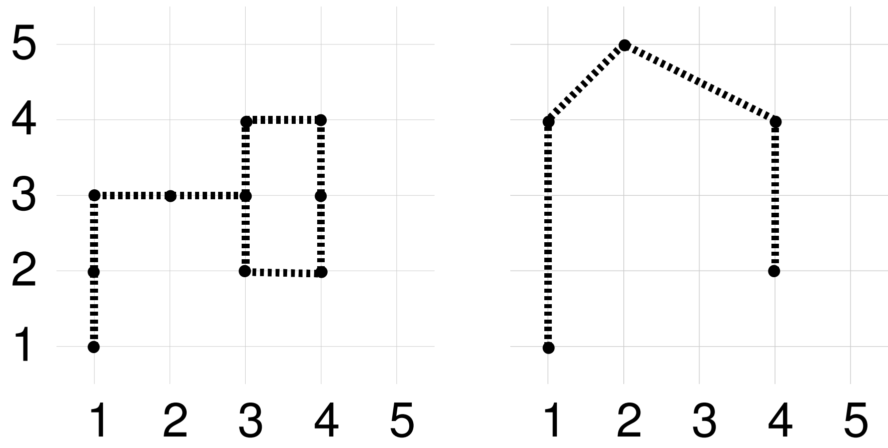

Figure 3.5.: On the left, a valid path $\langle(1,1),(1,2),(1,3),(2,3),(3,3),(3,4),(4,4),(4,3),$ $(4,2),(3,2),(3,3)\rangle$ is shown. The set on the left is not a path according to the definition above, since the points are not nearby.

transport and vice versa, we can choose a transport fitting our requirements. An acceptable choice could be to bind the parallel transport to a nearby element to the left action of the corresponding Schwinger element $S_{(1,0)} = \mathcal{U}$, $S_{(-1,0)} = \mathcal{U}^{-1}$, $S_{(0,1)} = \mathcal{V}$ respectively $S_{(0,-1)} = \mathcal{V}^{-1}$. We call the transports in the two lattice directions Γ_q and Γ_p according to the meaning of the directions in QPS:

$$\begin{aligned}\Gamma_q : ((r,s),S_{\mathbf{t}}) \mapsto & L_{\mathcal{U}}\left((r+1,s),S_{\mathbf{t}}\right) \\ &= L_{S_{(1,0)}}\left((r+1,s),S_{\mathbf{t}}\right) \\ &= \Big((r+1,s),S_{(1,0)}S_{\mathbf{t}}\Big) \\ &= \Big((r+1,s),e^{-i\alpha_2((1,0),\mathbf{t})}S_{(t_1+1,t_2)}\Big) \\ &= \Big((r+1,s),e^{-i\frac{\pi}{N}t_2}S_{(t_1+1,t_2)}\Big) \qquad (3.47)\\ \Gamma_p : ((r,s),S_{\mathbf{t}}) \mapsto & L_{S_{(0,1)}}\left((r,s+1),S_{\mathbf{t}}\right) \\ &= \Big((r,s+1),S_{(0,1)}S_{\mathbf{t}}\Big) \\ &= \Big((r,s+1),e^{i\frac{\pi}{N}t_1}S_{(t_1,t_2+1)}\Big) \qquad (3.48)\end{aligned}$$

Notice the symmetry on the Schwinger basis elements in contrast to plain $\mathcal{U}^m\mathcal{V}^n$ products, which produce a phase factor when being multiplied with $\mathcal{V}$ but not with $\mathcal{U}$. The dependency of the factors on the actual element will not cause any problems in the following, as we will see in the holonomy section.

It is easy to see that the transport in negative direction is just the inverse of the one in positive direction since the index appearing in the phase factor is not altered:

$$\begin{aligned}
\Gamma_q^{-1} : ((r,s), S_{\mathbf{t}}) \mapsto & \Big((r-1,s), \mathcal{U}^{-1} S_{\mathbf{t}}\Big) \\
& = L_{S_{(-1,0)}}((r-1,s), S_{\mathbf{t}}) && (3.49) \\
\Gamma_p^{-1} : ((r,s), S_{\mathbf{t}}) \mapsto & \Big((r,s-1), \mathcal{V}^{-1} S_{\mathbf{t}}\Big) \\
& = L_{S_{(0,-1)}}((r,s-1), S_{\mathbf{t}}) && (3.50) \\
& && (3.51)
\end{aligned}$$

Having chosen the left action, we ensured that the parallel transport commutes with the right action of the bundle,

$$\Gamma_{p/q} R_g e = R_g \Gamma_{p/q} e, \tag{3.52}$$

for any $g \in G$ and $e \in E$ since left and right action always commute due to the associativity of the group.

We can now make the following

Definition 16 *A lift is called a **horizontal lift** if and only if its elements follow the parallel transport defined above.*

It is easy to see that horizontal lifts according to this definition have some very similar properties to the ones on continuous base spaces.

Given a path P, there is an equivalence class of horizontal lifts $[\tilde{P}]$. If additionally a single point in the image of the path is given, the lift is uniquely defined since one only has to parallel-transport the point along the path in the base space to construct the complete lift.

Having defined horizontal lifts and parallel transport, we may proceed with studying holonomy on these bundles.

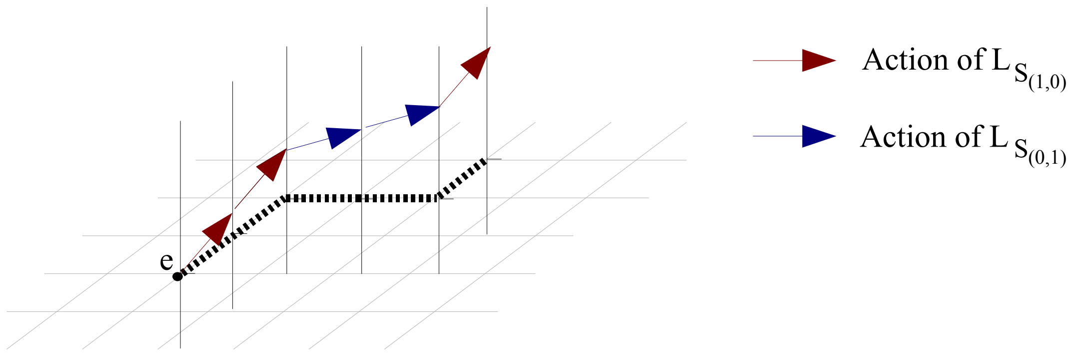

Figure 3.6.: A point e in $\pi^{-1}(P)$ is transported along the path by Γ_q and Γ_p constituting the horizontal lift $\tilde{P}$ of P through e

3.3.3. Holonomy

Since the base space is discrete, there is a smallest possible cycle (except for the "constant path" that only consists of its starting point):

$$C = \langle (m,n),(m+1,n),(m+1,n+1),(m,n+1),(m,n)\rangle$$

It is easy to see that the values of m and n do not effect the parallel transport and can thus be left arbitrary. An element e with $\pi(e) = (m,n)$ constitutes a unique horizontal lift $\tilde{C}$ of C by defining a starting point. The ending point is then

$$\begin{aligned} e' &= \Gamma_p^{-1}\Gamma_q^{-1}\Gamma_p\Gamma_q e \\ &= \mathcal{V}^{-1}\mathcal{U}^{-1}\mathcal{V}\mathcal{U}e \\ &= \omega^1\mathcal{V}^{-1}\mathcal{V}\mathcal{U}^{-1}\mathcal{U}e \\ &= \omega^1 e \end{aligned} \tag{3.53}$$

It is quite obvious that the same factor would occur if we had started into another direction of the lattice. The only thing that matters is the orientation in which we circle the elements. If we change the orientation of the path from negative (clockwise) to positive (counter-clockwise), we get

$$\begin{aligned} e' &= \mathcal{V}\mathcal{U}^{-1}\mathcal{V}^{-1}\mathcal{U}e \\ &= \omega^{-1}\mathcal{V}\mathcal{V}^{-1}\mathcal{U}^{-1}\mathcal{U}e \\ &= \omega^{-1}e \end{aligned} \tag{3.54}$$

What happens if we consider wider cycles, for example

$\langle (m,n),(m+1,n),(m+2,n),(m+2,n+1),(m+1,n+1),(m,n+1),(m,n)\rangle$ or
$\langle (m,n),(m+1,n),(m+1,n+1),(m,n+1),(m,n),(m,n-1),$
$\quad (m-1,n-1),(m-1,n),(m,n)\rangle$

(see figure 3.7)?

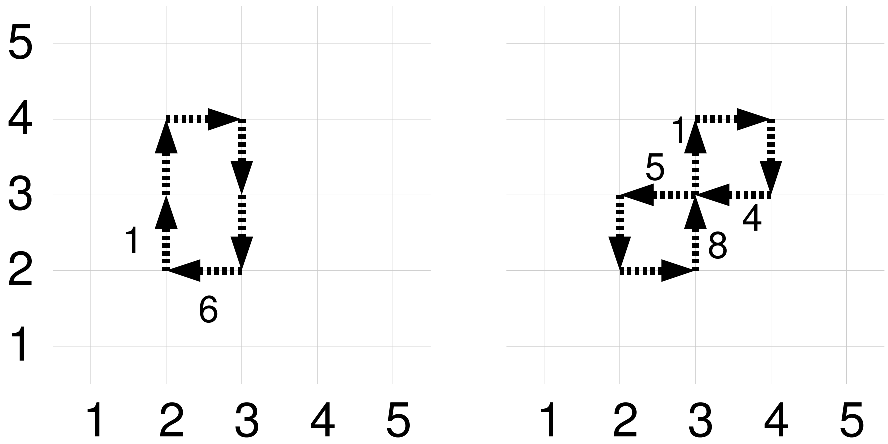

Figure 3.7.: Two wider cycles. The first one is a negatively oriented cycle enclosing 2 lattice cells, the other one is a combination of two cycles, one being positively, the other negatively oriented.

The first one has obviously completely negative orientation and gives

$$\begin{aligned} e' &= \mathcal{V}^{-1}\mathcal{U}^{-1}\mathcal{U}^{-1}\mathcal{V}\mathcal{U}\mathcal{U}e \\ &= \omega^2 e. \end{aligned} \tag{3.55}$$

The second one has negative orientation in the first cycle and positive in the second. It can simply be divided into two cycles giving

$$\begin{aligned} e' &= \mathcal{U}\mathcal{V}\mathcal{U}^{-1}\mathcal{V}^{-1}\cdot\mathcal{V}^{-1}\mathcal{U}^{-1}\mathcal{V}\mathcal{U}e \\ &= \omega^{-1}\cdot\omega^{1}e \\ &= e \end{aligned} \tag{3.56}$$

When considering wider cycles, the automorphism $\tau[\tilde{C}]$ on $\pi^{-1}(m,n)$ defined by the cycle $\tilde{C}$ just adds the factor ω to the power of the number of phase space cells enclosed in negative direction minus the the number of phase space cells enclosed in positive direction in the cycle.

Obviously this automorphism commutes with the right action - on the one hand since it is generated by the left action, on the other hand since it is always an element of the Abelian subgroup of G generated by the complex number ω.

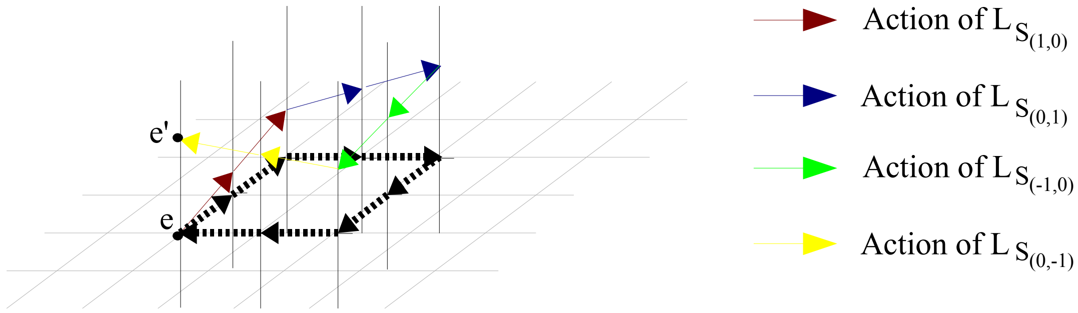

Figure 3.8.: The lift $\tilde{C}$ of a given cycle C defines an automorphism $\tau[\tilde{C}] : e \mapsto e'$

Consequently, the holonomy group of an arbitrary point in the bundle is the finite dimensional group

$$\Phi_T = \{\omega^n | n \in \mathbb{Z}_N\} \tag{3.57}$$

Let us have a look at the factor of a minimal cycle again. It is

$$\omega^1 = \omega = e^{2\pi i/N}$$

for a negative cycle and

$$\omega^{-1} = e^{-2\pi i/N}$$

for a positive one.

We can introduce the symplectic form α_2, by identifying these factors with

$$e^{-2i\alpha_2(\mathbf{a},\mathbf{b})} \tag{3.58}$$

where **a** is the outgoing and **b** is the incoming difference vector on the lattice, e.g. $(1,0)$ as the difference between $(m+1,n)$ and (m,n).

This is not quite surprising since the 2-form measures (half, which explains the factor 2) the oriented area spanned by the two vectors, which is closely related to the holonomy. This formula not only works with minimal cycles but for every rectangular cycle, in which case **a** and **b** are the difference vectors between the corners of the cycle. In contrast, it does not work with other shapes since in this case, the incoming and outgoing vectors do not carry the full information. We can, however, divide the cycle into smaller rectangular cycles (see figure 3.9). The choice of the corner of the particular cycle is then arbitrary and the results of the symplectic forms have to be added or, alternatively, the phase factors to be multiplied.

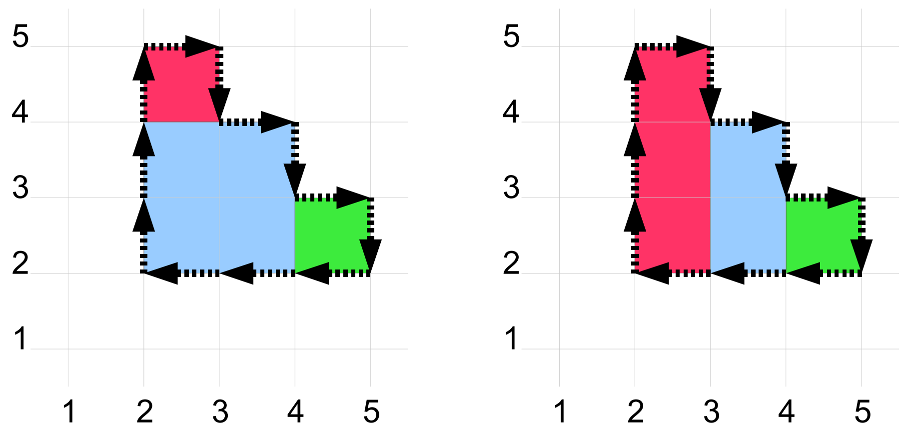

Figure 3.9.: Cycles can be divided into arbitrary rectangular "subcycles", since only orientation and enclosed area matter.

Indeed this brings to mind the connection between the Lie algebra of the holonomy group and the symplectic curvature form of common principal bundles, which is described by the Ambrose-Singer theorem (theorem 5).

3.3.4. Some words on the continuum limit

Let us start with a rough examination of the limit. Will the bundle stay curved when N approaches ∞ or will it flatten? Consider a cycle that keeps "approximately the same size" when N grows, e.g. the path that circulates the square of $\lfloor\sqrt{N}\rfloor \times \lfloor\sqrt{N}\rfloor$ cells. Our former results say that the phase factor induced by a parallel transport along this path is $e^{-2i\alpha_2((\lfloor\sqrt{N}\rfloor,0),(0,-\lfloor\sqrt{N}\rfloor))} = e^{2\pi i\frac{\lfloor\sqrt{N}\rfloor^2}{N}}$. Due to the rounding, $\lfloor\sqrt{N}\rfloor^2$ will mostly be a bit short, so the quotient will not be 1, but since the square numbers grow quadratically, while the distances between them grow linearly, the larges "outlier" will approach 1. Consequently, we can expect a curved bundle even in the limit.

We have already discussed the continuum limit of the base space in subsection 2.3.2. The Heisenberg group of the fiber becomes a Lie group of dimension 3, since an element is completely defined through the now continuous powers of the components $e^i, \mathcal{U}$ and $\mathcal{V}$.

A single element g of the group has the form

$$g = e^{i\left(\frac{a_1 a_2}{2} + a_3\right)} e^{i a_1 \hat{q}} e^{i a_2 \hat{p}}. \tag{3.59}$$

with $a_1, a_2 \in \mathbb{R}$, $a_3 \in [0, 2\pi]$.

We will denote the full continuous bundle as P, the fiber as G and the base manifold as M in the following.

The form of the difference vectors has been presented in equations (3.45) and (3.46). In the continuum limit these three vectors become differential quotients – the derivations after the 3 parameters at the point – spanning the 3-dimensional tangent space of the particular point.

The continuous version of α has been calculated in equation (2.145):

$$\alpha_2(\mathbf{a}, \mathbf{b}) = \frac{a_1 b_2 - a_2 b_1}{2} \tag{3.60}$$

If we call one coordinate function of the 2-dimensional Euclidian plane q and the other one p, we can express this form in the basis of the differentials of these functions. It is just

$$\alpha_2 = dq \wedge dp, \tag{3.61}$$

thus a very basic quantity. It acts of course on the tangent space of the base manifold, which is also $\mathbb{R}^2$ and – due to the fact that the base manifold is flat and the bundle is trivial – may easily be embedded into the tangent space of the whole bundle.

Let us regard the full continuous bundle as a manifold. The inverses of the coordinate function ϕ_i are well known as

$$\phi_i^{-1} : (q, p, a_1, a_2, a_3) \mapsto \left(q, p, e^{i\left(\frac{a_1 a_2}{2} + a_3\right)} e^{i a_1 \hat{q}} e^{i a_2 \hat{p}}\right) \tag{3.62}$$

where q, p, a_1, a_2 may be in $\mathbb{R}$ for every open neighborhood, while we have to choose different neighborhoods to respect the periodicity of the complex exponential function. For a_3 it is obvious that we need at least 2 intervals, for example $(0, 2\pi)$ and $(-\pi, \pi)$.

Consequently, the tangential space is spanned by the canonical coordinate basis, which in this case is $\{\partial_q, \partial_p, \partial_{a_1}, \partial_{a_2}, \partial_{a_3}\}$.

The representation in a point with coordinates (Q, P, A_1, A_2, A_3) in the surrounding space can be easily calculated:

$$\partial_q = (1, 0, \hat{0}) \tag{3.63}$$
$$\partial_p = (0, 1, \hat{0}) \tag{3.64}$$
$$\partial_{a_1} = \left(0, 0, i\frac{A_2}{2} e^{i\left(\frac{A_1 A_2}{2} + A_3\right)} e^{iA_1\hat{q}} e^{iA_2\hat{p}} + i e^{i\left(\frac{A_1 A_2}{2} + A_3\right)} \hat{q} e^{iA_1\hat{q}} e^{ia_2\hat{p}}\right) \tag{3.65}$$
$$\partial_{a_2} = \left(0, 0, i\frac{A_1}{2} e^{i\left(\frac{A_1 A_2}{2} + A_3\right)} e^{iA_1\hat{q}} e^{iA_2\hat{p}} + i e^{i\left(\frac{A_1 A_2}{2} + A_3\right)} e^{iA_1\hat{q}} \hat{p} e^{ia_2\hat{p}}\right) \tag{3.66}$$
$$\partial_{a_3} = \left(0, 0, i e^{i\left(\frac{A_1 A_2}{2} + A_3\right)} e^{iA_1\hat{q}} e^{iA_2\hat{p}}\right) \tag{3.67}$$

In the point $p_0 = (0, 0, \hat{\mathbb{1}})$ with the coordinates $(0, 0, 0, 0, 0)$, this is just

$$\partial_q = (1, 0, \hat{0}) \tag{3.68}$$
$$\partial_p = (0, 1, \hat{0}) \tag{3.69}$$
$$\partial_{a_1} = (0, 0, i\hat{q}) \tag{3.70}$$
$$\partial_{a_2} = (0, 0, i\hat{p}) \tag{3.71}$$
$$\partial_{a_3} = (0, 0, i\hat{\mathbb{1}}) \tag{3.72}$$

Due to the local (and in this case even global) triviality of the bundle, $T_{p_0}P = T_{(0,0)}M \times T_{\hat{\mathbb{1}}}G$. Consequently, the tangent space of G in $\hat{\mathbb{1}}$ is spanned by $\{i\hat{q}, i\hat{p}, i\hat{\mathbb{1}}\}$. This space is isomorphic to the Lie algebra of the group $\mathcal{L}(G)$.

Let us further examine the parallel transport. In the discrete case, the parallel transported nearby points of an arbitrary point $p_{0,0} = \Big((m, n), \omega^r S_{(s,t)}\Big)$
$= \Big((m, n), e^{2i\frac{\pi}{N}r} e^{\frac{\pi i}{N}st} e^{i\sqrt{2\pi/N}s\hat{q}} e^{i\sqrt{2\pi/N}t\hat{p}}\Big)$ in the bundle are

$$p_{1,0} = \Big((m+1, n), e^{2i\frac{\pi}{N}r} S_{(1,0)} S_{(s,t)}\Big)$$

$$= \left((m+1,n), e^{2i\frac{\pi}{N}r} e^{-i\frac{\pi}{N}t} e^{i\frac{\pi}{N}(s+1)t} e^{i\sqrt{2\pi/N}(s+1)\hat{q}} e^{i\sqrt{2\pi/N}t\hat{p}}\right) \tag{3.73}$$

$$= \left((m+1,n), e^{2i\frac{\pi}{N}r} e^{i\frac{\pi}{N}st} e^{i\sqrt{2\pi/N}(s+1)\hat{q}} e^{i\sqrt{2\pi/N}t\hat{p}}\right) \tag{3.74}$$

$$p_{-1,0} = \left((m-1,n), e^{2i\frac{\pi}{N}r} S_{(-1,0)} S_{(s,t)}\right)$$

$$= \left((m-1,n), e^{2i\frac{\pi}{N}r} e^{i\frac{\pi}{N}st} e^{i\sqrt{2\pi/N}(s-1)\hat{q}} e^{i\sqrt{2\pi/N}t\hat{p}}\right) \tag{3.75}$$

$$p_{0,1} = \left((m,n+1), e^{2i\frac{\pi}{N}r} S_{(0,1)} S_{(s,t)}\right)$$

$$= \left((m,n+1), e^{2i\frac{\pi}{N}r} e^{i\frac{\pi}{N}s} e^{i\frac{\pi}{N}s(t+1)} e^{i\sqrt{2\pi/N}s\hat{q}} e^{i\sqrt{2\pi/N}(t+1)\hat{p}}\right)$$

$$= \left((m,n+1), e^{2i\frac{\pi}{N}r} e^{i\frac{\pi}{N}st} e^{2i\frac{\pi}{N}s} e^{i\sqrt{2\pi/N}s\hat{q}} e^{i\sqrt{2\pi/N}(t+1)\hat{p}}\right) \tag{3.76}$$

$$p_{0,-1} = \left((m,n-1), e^{2i\frac{\pi}{N}r} S_{(0,-1)} S_{(s,t)}\right)$$

$$= \left((m,n-1), e^{2i\frac{\pi}{N}r} e^{i\frac{\pi}{N}st} e^{i\sqrt{2\pi/N}s\hat{q}} e^{i\sqrt{2\pi/N}(t-1)\hat{p}}\right) \tag{3.77}$$

We can regard the difference vectors as difference quotients if we divide them by the distance of the lattice points and substitute the position on the lattice with their actual coordinates so that the distances cancel each other in the lattice components:

$$\frac{p_{1,0} - p_{0,0}}{\sqrt{2\pi/N}} = \left((1,0), e^{2i\frac{\pi}{N}r} e^{\frac{\pi i}{N}st} e^{i\sqrt{2\pi/N}s\hat{q}} \left(\frac{e^{i\sqrt{2\pi/N}\hat{q}} - 1}{\sqrt{2\pi/N} - 0}\right) e^{i\sqrt{2\pi/N}t\hat{p}}\right) \tag{3.78}$$

The limit – as N approaches ∞ – is a tangent vector which we defined as horizontal formerly. Since it is just the differential quotient it can easily be calculated:

$$v_q = \lim_{N\to\infty} \frac{p_{1,0} - p_{0,0}}{\sqrt{2\pi/N}}$$

$$= \left((1,0), e^{ia_3} e^{i\frac{a_1 a_2}{2}} e^{ia_1\hat{q}} \left(\frac{d}{dx} e^{ix\hat{q}}|_{x=0}\right) e^{ia_2\hat{p}}\right)$$

$$= \left((1,0), ie^{ia_3} e^{i\frac{a_1 a_2}{2}} e^{ia_1\hat{q}} \hat{q} e^{ia_2\hat{p}}\right) \tag{3.79}$$

We can perform the very same operations with the nearby points in the other direction. The opposite point will lead to the same result as expected. The other lattice direction will give

$$v_p = \lim_{N\to\infty} \frac{p_{0,1} - p_{0,0}}{\sqrt{2\pi/N}}$$

$$
\begin{aligned}
&= \lim_{N\to\infty}\left((0,1), e^{2i\frac{\pi}{N}r}e^{i\frac{\pi}{N}st}e^{i\sqrt{2\pi/N}s\hat{q}}\frac{e^{i2\frac{\pi}{N}s}e^{i\sqrt{2\pi/N}\hat{p}}-1}{\sqrt{2\pi/N}-0}e^{i\sqrt{2\pi/N}t\hat{p}}\right)\\
&= \left((0,1), e^{ia_3}e^{i\frac{a_1a_2}{2}}e^{ia_1\hat{q}}\left(\frac{d}{dx}\{e^{ix^2}e^{ix\hat{p}}\}|_{x=0}\right)e^{ia_2\hat{p}}\right)\\
&= \left((0,1), ia_1e^{ia_3}e^{i\frac{a_1a_2}{2}}e^{ia_1\hat{q}}e^{ia_2\hat{p}}+ie^{ia_3}e^{i\frac{a_1a_2}{2}}e^{ia_1\hat{q}}\hat{p}e^{ia_2\hat{p}}\right). \qquad (3.80)
\end{aligned}
$$

Since we did not allow other direct steps on the lattice, these are the only horizontal vectors we are able to calculate. However, these are enough to span a space that we may call horizontal space since the horizontal space in the common theory has the same dimension as the base space, which is obviously also 2. Consequently, if we had allowed other directions, they would either provide linearly dependent vectors or break the concept. Indeed, the latter is the case, so we are confirmed in the construction of your discrete theory.

If we analyze $v_q \in T_xM \times T_{\mathbb{1}}G$ – that is in the point where $a_1 = a_2 = a_3 = 0$ – we find that is just the anti-hermitian operator

$$
\begin{aligned}
v_q &= ((1,0), i\hat{q})\\
v_p &= ((0,1), i\hat{p})
\end{aligned}
\qquad (3.81)
$$

We can compare these vectors to equations (3.63)-(3.67) and get:

$$v_q = \partial_q + \partial_{a_1} - \frac{A_2}{2}\partial_{a_3} \qquad (3.82)$$

$$v_p = \partial_p + \partial_{a_2} + \frac{A_1}{2}\partial_{a_3} \qquad (3.83)$$

Since we now have the representation of horizontal vectors, we can calculate the $\mathcal{L}(G)$-valued connection form in terms of the dual basis $\{dq, dp, da_1, da_2, da_3\}$.

Its general form is

$$\omega = \alpha dq + \beta dp + \gamma_1 da_1 + \gamma_2 da_2 + \gamma_3 da_3 \qquad (3.84)$$

We need 5 equations for a unique solution. The first two are provided by the demand of definition 8 claiming that the horizontal space is the kernel of the 1-form:

$$\omega(v_q) \stackrel{!}{=} 0 \Rightarrow 0 = \alpha + \gamma_1 - \frac{A_2}{2}\gamma_3 \tag{3.85}$$

$$\omega(v_p) \stackrel{!}{=} 0 \Rightarrow 0 = \beta + \gamma_2 + \frac{A_1}{2}\gamma_3 \tag{3.86}$$

The 3 equations left are provided by property 2 of the same definition:

$$\gamma_1 = i\hat{q} \tag{3.87}$$

$$\gamma_2 = i\hat{p} \tag{3.88}$$

$$\gamma_3 = i\hat{\mathbb{1}} \tag{3.89}$$

which leads to

$$\alpha = \frac{A_2}{2}i\hat{\mathbb{1}} - i\hat{q} \tag{3.90}$$

$$\beta = -\frac{A_1}{2}i\hat{\mathbb{1}} - i\hat{p} \tag{3.91}$$

Thus, the connection form is

$$\omega = \left(\frac{A_2}{2}i\hat{\mathbb{1}} - i\hat{q}\right)dq + \left(-\frac{A_1}{2}i\hat{\mathbb{1}} - i\hat{p}\right)dp + i\hat{q}da_1 + i\hat{p}da_2 + i\hat{\mathbb{1}}da_3 \tag{3.92}$$

It is now easy to calculate the curvature 2-form $\Omega = D\omega$:

$$\begin{aligned}\Omega(X,Y) &= D\omega(X,Y) = d\omega(X^H, Y^H) \\ &= i\frac{da_2 \wedge dq - da_1 \wedge dp}{2}(X^H, Y^H) \\ &= i\frac{da_2 \wedge dq + dp \wedge da_1}{2}(X^H, Y^H)\end{aligned} \tag{3.93}$$

These forms act on T_pP. The values lie in the subspace of the Lie algebra spanned by the $\mathbb{1}$ operator. We therefore omitted writing it down explicitly but treat the form as complex-valued. We notice that this result fits the Ambrose-Singer theorem stated above claiming that the values lie in the Lie algebra of the holonomy group.

The symplectic form α_2, however, acts on T_xM. To examine a connection between them we have to transport the just calculated forms to T_xM. This can be done using a pullback by a section. Since the bundle is trivial this can be achieved globally. The natural choice for a section in this treatise is of course the one defined by the Schwinger elements themselves:

$$\sigma_S : M \to P \tag{3.94}$$
$$\sigma_S : (\bar{q}, \bar{p}) \mapsto \left(\bar{q}, \bar{p}, e^{i\frac{\bar{q}\bar{p}}{2}} e^{i\bar{q}\hat{q}} e^{i\bar{p}\hat{p}} \right) \tag{3.95}$$

or in coordinate representation

$$x = (\bar{q}, \bar{p}) \mapsto (\bar{q}, \bar{p}, \bar{q}, \bar{p}, 0) = y \tag{3.96}$$

We bar the indices to distinguish between the coordinates in the base manifold and the first two coordinates of the bundle.

This section is globally continuous since the points in which problems may arise due to property 2 in section 2.2.2 disappear into infinity with the continuum limit.

The pullback f^* of an r-form $\eta \in \Omega^r_{f(x)} N$ by a map $f : M \to N$ is defined by

$$f^* \eta(X_1, \ldots, X_r) = \eta(f_* X_1, \ldots, f_* X_r) \tag{3.97}$$

where $X_1, \ldots, X_r \in T_xM$ and f_* is the differential map of f (see [Nak90] for this and the whole following procedure).

Let us calculate $\sigma_{S*}V$ for an arbitrary vector $T_xM \ni V = V^{\bar{q}} \partial_{\bar{q}} + V^{\bar{p}} \partial_{\bar{p}}$:

$$\begin{aligned}
\sigma_{S*}V &= V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \partial_{y^\alpha} \\
&= V^{\bar{q}} \Big(\frac{\partial y^1}{\partial \bar{q}} \partial_q + \frac{\partial y^3}{\partial \bar{q}} \partial_{a_1} \Big) + V^{\bar{p}} \Big(\frac{\partial y^2}{\partial \bar{p}} \partial_p + \frac{\partial y^4}{\partial \bar{p}} \partial_{a_2} \Big) \\
&= V^{\bar{q}} (\partial_q + \partial_{a_1}) + V^{\bar{p}} (\partial_p + \partial_{a_2}) \\
&= V^{\bar{q}} \Big(\partial_q + \partial_{a_1} - \frac{\bar{p}}{2} \partial_{a_3} \Big) + V^{\bar{p}} \Big(\partial_p + \partial_{a_2} - \frac{\bar{q}}{2} \partial_{a_3} \Big) + \left(\frac{V^{\bar{q}} \bar{p}}{2} + \frac{V^{\bar{p}} \bar{q}}{2} \right) \partial_{a_3} \\
&= V^{\bar{q}} v_q + V^{\bar{p}} v_p + \left(\frac{V^{\bar{q}} \bar{p}}{2} + \frac{V^{\bar{p}} \bar{q}}{2} \right) \partial_{a_3}
\end{aligned} \tag{3.98}$$

Thus

$$(\sigma_{S*}V)^H = V^{\bar{q}}v_q + V^{\bar{p}}v_p \tag{3.99}$$

Now we can just calculate the action of the pullback of the curvature 2-form Ω on two arbitrary vectors in T_xM:

$$\begin{aligned}
\sigma_S^*\Omega(V,W) &= \sigma_S^* d\omega(V^H, W^H) \\
&= i\frac{da_2 \wedge dq + da_1 \wedge dp}{2}(\sigma_{S*}V^H, \sigma_{S*}W^H) \\
&= \frac{i}{2}(V^{\bar{p}}W^{\bar{q}} - W^{\bar{p}}V^{\bar{q}} + V^{\bar{p}}W^{\bar{q}} - W^{\bar{p}}V^{\bar{q}}) \\
&= i(V^{\bar{p}}W^{\bar{q}} - W^{\bar{p}}V^{\bar{q}}) \\
&= -i(dq \wedge dp) \\
&= -i\alpha_2(V,W)
\end{aligned} \tag{3.100}$$

We see that – using the Schwinger section – the pullback is just (-i times) the symplectic form we found in chapter 2.

To finish with, we examine just another approach by determining the pullback of the connection 1-form by the section. This time, we calculate the form directly. Let $\omega = \omega_q dq + \omega_p dp + \omega_{a_1}da_1 + \omega_{a_2}da_2 + \omega_{a_3}da_3$ and $\sigma_S^*\omega = \bar{\omega}_{\bar{q}}d\bar{q} + \bar{\omega}_{\bar{p}}d\bar{p}$. The

$$\bar{\omega}_{\bar{q}} = \omega_\alpha \frac{\partial y^\alpha}{\partial q} = \omega_q + \omega_{a_1} = \frac{\bar{p}}{2}i\mathbb{1} + i\hat{q} - i\hat{q} = i\frac{\bar{p}}{2} \tag{3.101}$$

$$\bar{\omega}_{\bar{p}} = \omega_\alpha \frac{\partial y^\alpha}{\partial q} = \omega_p + \omega_{a_2} = -\frac{\bar{q}}{2}i\mathbb{1} + i\hat{p} - i\hat{p} = i\frac{\bar{q}}{2} \tag{3.102}$$

and

$$\sigma_S^*\omega = i\frac{\bar{p}d\bar{q} - \bar{q}d\bar{p}}{2} \tag{3.103}$$

When applying d on this 1-form

$$\begin{aligned}
d(\sigma_S^*\omega) &= \frac{i}{2}d\bar{p} \wedge d\bar{q} - \frac{i}{2}d\bar{q} \wedge d\bar{p} \\
&= -i\alpha_2.
\end{aligned} \tag{3.104}$$

we again get the symplectic form.

As we could see, the discrete theory developed in the former sections not only integrates seamlessly with the common continuous theory but even reproduces the symplectic form we expected.

As a last part of this treatise, we will have a short look at nontrivial phase spaces.

3.4. Nontrivial phase spaces

3.4.1. Definition of the Klein bottle and the Boy surface

Until now, we have always considered quite an easy phase space, the lattice torus $\mathbb{Z}_N \times \mathbb{Z}_N$. In this section we will briefly examine more complicated structures.

The torus is the result of gluing the edges of a plane together in the same orientation: exactly this is done by the identification of 0 and N in $\mathbb{Z}_N$.

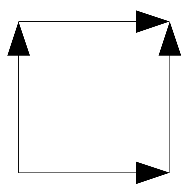

Figure 3.10.: A torus is created by gluing the edges of a plane together in the same orientation.

It is well known that we achieve topologically different results if we change the orientation of the edges. These are the Klein bottle and the real projective plane or

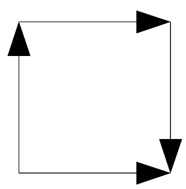

Figure 3.11.: Changing the orientation of a single edge results in the Klein bottle.

its immersion into $\mathbb{R}^3$, the Boy surface.

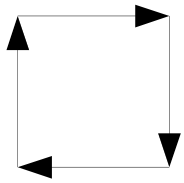

Figure 3.12.: Changing the orientation of two nearby edges results in the real projective plane.

In the previous section we already described the Klein bottle as a fiber bundle with base and fiber space S_1 itself. This case is different since we take the nontrivial structure into the basespace instead while the fibers stay the same. The difference between the bundle with the torus as a base space lies in the way the edges of the plane are glued together. Consequently, we have to reconsider paths and transports that cross the boundary of the lattice.

We first need to define the base space. While $\mathbb{Z}_n$ can be defined as the the set of the N equivalence classes

$$[n] = \{n + kN | k \in \mathbb{Z}\} \tag{3.105}$$

the case is slightly more complicated now.

A step in one direction may effect the value of the other direction. For example,

$$\langle (0, N-2), (0, N-1), (N-1, 0) \rangle \tag{3.106}$$

is a valid path on these nontrivial spaces (if we assume the corresponding edge of the Klein bottle to be twisted). In contrast

$$\langle (0, N-2), (0, N-1), (0, 0) \rangle \tag{3.107}$$

is no longer valid, since there is only an edge connecting $(0, N-1)$ with $(N-1, 0)$ instead of $(0, 0)$.

Thus, we have to describe the sets through a single equivalence class respectively. For the Klein bottle we can define an equivalence relation $\sim_K$ by

$$\begin{aligned}(m,n) \sim_K (r,s) \Leftrightarrow\ & \exists k,l \in \mathbb{Z} : ((m,n) = (r + lN, 2kN + s) \\ & \vee\ (m,n) = (N-1-r+lN, (N-1-s) - (2k+1)N)) \\ \Leftrightarrow\ & \exists k \in \mathbb{Z} : ((m,n) = (r + lN, 2kN + s)\end{aligned}$$

$$\vee \ (m,n) = (lN - 1 - r, 2kN - (s+1)) \tag{3.108}$$

The lattice Klein bottle is then the set of the N^2 equivalence classes defined by this relation. We can do the same thing with the lattice Boy surface:

$$\begin{aligned} (m,n) \sim_B (r,s) \Leftrightarrow \exists k,l \in \mathbb{Z} : ((m,n) &= (2lN + r, 2kN + s) \\ \vee\ (m,n) &= (2lN + r, 2kN - s - 1) \\ \vee\ (m,n) &= (2lN - r - 1, 2kN + s) \\ \vee\ (m,n) &= (2lN - r - 1, 2kN - s - 1)) \end{aligned} \tag{3.109}$$

Again, the N^2 equivalence classes form the discrete Boy surface.

Figure 3.13.: A lattice model of the Boy surface located at the Mathematisches Forschungsinstitut Oberwolfach library building ([Wei])

How does this affect a possible path? Consider the discrete Klein bottle with lattice dimension N. Then

$$\langle (n, N-2), (n, N-1), (N-n-1, 0) \rangle \tag{3.110}$$

is a valid path, while

$$\langle (n, N-2), (n, N-1), (n, 0) \rangle \tag{3.111}$$

is not in general valid (unless $N = 2n+1$), as $(n, N-1)$ is not nearby $(n, 0)$ anymore.

The discrete real Boy surface is even more complicated. Here

$$\langle (0, N-2), (N-1, 1), (N-1, 0), (0, N-1), (0, N-2) \rangle \tag{3.112}$$

is a valid cycle.

3.4.2. Holonomy on nontrivial phase spaces

Let us examine how this topology affects the holonomy groups Φ_K and Φ_B of Klein bottle and Boy surface.

Obviously Φ_T is a subgroup of both groups since all its elements can be achieved without crossing the glued-together boundaries of the lattice. But this is not the complete group. When crossing the boundary, the connection between the change of the lattice point and the change of an arbitrary Schwinger element will break. For example, if we parallel transport $S_{(N-1,0)}$ on $(N-1,0)$ one step in the q direction of a Klein bottle, the lattice point will be $(0, N-1)$, but the fiber element will be $S_{0,0}$. If we parallel transport this element back to the starting point without crossing boundaries, it will differ from the starting point by the left action of a group element of the form $\omega^k S_{(0,-1)}$, where k is an arbitrary number in $\mathbb{Z}_N$ depending on the cycle taken.

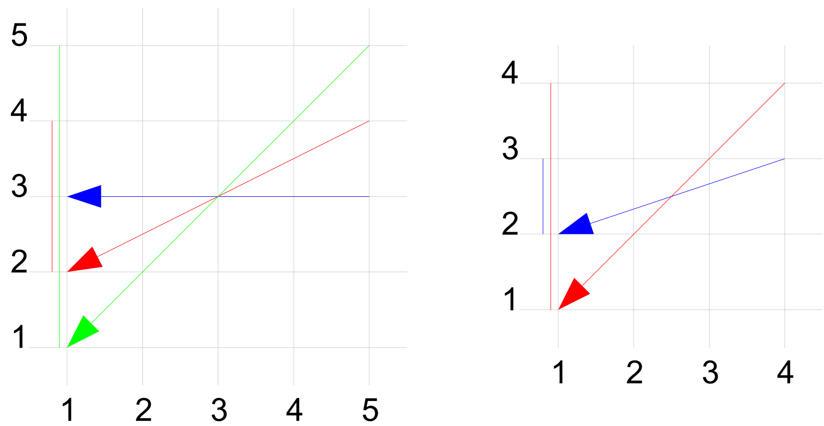

Figure 3.14.: In twisted phase spaces with even lattice dimensions, odd numbers of lattice points are skipped when transporting a point over the boundary and even numbers are skipped, if the lattice dimension is odd.

More generally speaking, a single crossing of a boundary may result in a difference of $S_{(0,2k+1)}$ for even N or $S_{(0,2k)}$ for odd N (see figure 3.14).

Since in odd dimensions $\lceil \frac{N}{2} \rceil$ border crossings result in a shift by 1, in both cases the holonomy group Φ_K is now generated by the elements $\{\omega, S_{(1,0)}\}$ or $\{\omega, S_{(0,1)}\}$

depending on which component of the phase space is twisted.

It is now easy to see that the generators of Φ_B are $\{\omega, S_{(1,0)}, S_{(0,1)}\}$. Consequently it equals the full Heisenberg group G of the fiber.

3.4.3. Some words on the continuum limit

The considered nontrivial phase spaces differ from the torus in the connection of the lattice boundaries. If we use the limit described in section 2.3.2 - namely the cell size $2\pi/N$ - these boundaries will vanish into infinity when considering the continuous case since the total scale of the lattice grows with $\sqrt{N}$.

If we want to maintain the nontrivial structure, we have to keep the size by defining the cell length proportional to $1/N$ instead of $1/\sqrt{N}$. For example, we may choose $\sqrt{2\pi}/N$. The full lattice space stays compact then and covers a total area of 2π independent of the lattice dimension.

Numbers of the form $\sqrt{2\pi}/N \cdot m$ become real numbers a in the interval $(0, 2\pi)$ or $(-\pi, \pi)$ depending on how we choose the elements of the circular group. Sums become integrals:

$$\sqrt{2\pi}/N \sum_{m=0}^{N-1} \longrightarrow \int_{(0,2\pi)} da$$

Unfortunately, this is not the physically interesting limit for a single particle problem. Nevertheless, there are physically interesting systems with compact phase spaces. A further examination of the spaces presented – including the calculation of connection and curvature forms – should be worth a try. We may expect a more complicated situation since due to the Ambrose-Singer theorem the values of these forms should span not only the imaginary numbers but a larger subspace of the full Lie algebra, because we found that Schwinger elements are now part of the holonomy group.

4. Conclusion

We will end this thesis with a resume about what we did, and found, and what needs further examination.

4.1. Summary

In the first part we considered the quantum phase space in terms of noncommutative differential geometry. Following relevant literature, especially [KDVM90b], [Mad99] and others which are named in the particular places of the text, a short introduction to vector fields and differential forms on the differential vector space $M_N(\mathbb{C})$ was given. Special emphasis has been laid on the construction of a canonical symplectic form analogous to the one known from classical mechanics.

The canonical choice of this form has been shown to be just the (scaled) commutator of two matrices. Essentially following ideas from [Ald01], it has been examined using a special unitary basis, the Schwinger basis. The elements of this basis commute with a phase factor,

$$S_{\mathbf{i}} S_{\mathbf{j}} = e^{-2i\alpha_2(\mathbf{i},\mathbf{j})} S_{\mathbf{j}} S_{\mathbf{i}} \tag{4.1}$$

and multiply satisfying

$$S_{\mathbf{i}} S_{\mathbf{j}} = e^{-i\alpha_2(\mathbf{i},\mathbf{j})} S_{\mathbf{i}+\mathbf{j}} \tag{4.2}$$

where α_2 is a symplectic form on the discrete "quantum phase space" $\mathbb{Z}_N \times \mathbb{Z}_N$. Extended with the discrete group of these complex numbers, they form a Heisenberg group.

[Ald01] proposes an approach to find a nontrivial solution to the Yang-Baxter equation. Therefore, in an excursus the concepts of braiding and braiding groups, which basically describe permutation of elements maintaining a "history", and their connection to the Yang-Baxter equation have been introduced. In this treatise, it could be shown, using analytical and computer algebraic methods that this ansatz to find a solution by braiding Schwinger elements – keeping a "history" of the permutation through the phase factors – does not lead to the results expected, but fails due to quite basic problems. Furthermore, it could be shown that not only the ansatz provided by Aldrovandi collapses but also that every other ansatz based on the braiding of Schwinger elements has to fail.

Using the Schwinger basis, the symplectic form derived in the first sections has been further examined by calculating concrete expressions for products of general matrices and their commutators which are, as we remember, just the symplectic form. An important role in these expressions is played by the symplectic form α_2.

Multiplication of Schwinger elements leads to a phase factor and the addition of the indices. One could view a Schwinger element as being attached to the corresponding point in the lattice of quantum phase space. Then the left action of another Schwinger element could be viewed as a transport on this lattice. Since it leaves a phase factor, closed walks on the lattice may not finish in the exact starting point.

All this is very similar to the theory of connections on fiber bundles. The difference to the common theory is discreteness of the base space in the case on hand. In the second part of this thesis this structure has been examined further. Firstly, following the pertinent literature such as [Ste74] and [CBDM82], an introduction to the well-known continuous theory has been given, including the general topological structure of fiber bundles and connections on principal bundles.

Subsequently, a discrete analog to the continuous theory has been developed, in which the lattice of the quantum phase space forms the base space, and the Heisenberg group including the Schwinger elements is identified with the fiber space. Based on the parallel transport through Schwinger elements mentioned above and on graph theory, properties of the differentiable case have been redefined: Viewing the lattice of the quantum phase space as a graph, curves become paths, and lifts are defined in the the preimage of the projection. The parallel transport constitutes which lifts are horizontal. In the following the resulting holonomy has been examined. It could be shown that in the trivial case of the lattice torus $\mathbb{Z}_N \times \mathbb{Z}_N$ the holonomy group is generated just by the symplectic form α_2 derived in the first part. This seems to be related to the connection between the curvature form on common principal bundles and their holonomy group, which is given by the Ambrose-Singer theorem stated in the introduction.

In the continuum limit it could be shown that the discrete theory seamlessly passed into the commonly known continuous theory of connection forms on fiber bundles. The connection form and its exterior covariant derivation, the curvature form, have been calculated. It has been found that the curvature form can even be pulled back to the symplectic form α_2 by the section defined by the Schwinger elements.

In more complicated topologies, some of these results become more complicated. Lattice versions of the Klein bottle and the Boy surface have been defined and examined using the methods developed before with the result that more complicated base spaces extend the holonomy group up to the full Heisenberg group. The continuum limit of the bundle theory thus can be expected to be more complicated since the forms will have values no longer isomorphic to the complex values of the symplectic form.

4.2. Outlook

Due to the fact that this treatise has been created outside of the research field of the local group, and opens at least in the second part – as far as the author knows – a new field with the description of connections on discrete fiber bundles, we could not tie in with existing works. Consequently quite a lot of questions concerning the context had to be left open in the scope of this diploma thesis.

In the first part we followed results from [KDVM90b]. After having developed a symplectic form in the noncommutative matrix differential geometry, we switched over to another rather mathematical topic, the discrete fiber bundle theory. We left open the actual connection between the semiclassical limit of the quantum symplectic form and the classical one. A detailed examination if and how the limit passes into the classical theory is owing.

The discrete fiber bundle theory works as far as we used it. A rough examination of the continuum limit shows that it fits the common continuous theory. Nevertheless, in the actual bundle we considered, the fibers could be embedded into the space of $N \times N$ matrices in the finite case. However, the embedding in the continuous case had to be left open and demands another mathematically more elaborate examination.

In the last chapter, we considered nontrivial phase spaces. Due to the extended set of values of the curvature form, problems with the identification of the pullback with the symplectic form can be expected. Further examination is needed.

The choice of these nontrivial spaces has been made especially because it is topologically obvious to glue the boundaries together in another orientation. A physical meaning is still to be found. A further approach is to examine another compact phase space whose physical meaning is already known. An example would be to transport the developed theories to the spherical phase space of a single spin system.

A. Appendix

A.1. Mathematica program for Aldrovandi solution of braid equation

Aldrovandi braid solution from Schwingerbasis

Definition of functions depending on indices and dimension.

```
α2[m1_, m2_, n1_, n2_, d_] = π / d (m1 n2 - m2 n1);
B[i1_, i2_, j1_, j2_, m1_, m2_, n1_, n2_, d_] =
 KroneckerDelta[Mod[i1 + j1, d], Mod[m1 + n1, d]]
  KroneckerDelta[Mod[i2 + j2, d], Mod[m2 + n2, d]]
  Exp[ⅈ (α2[m1, m2, n1, n2, d] - α2[i1, i2, j1, j2, d])] ;
```

Left and right sium of braid equation

```
BSumL[i1_, i2_, j1_, j2_, k1_, k2_, m1_, m2_, n1_, n2_, r1_, r2_, d_] :=
 Sum[B[k1, k2, j1, j2, a1, a2, b1, b2, d] B[b1, b2, i1, i2, c1, c2, r1, r2, d]
   B[a1, a2, c1, c2, m1, m2, n1, n2, d], {a1, 0, d - 1}, {a2, 0, d - 1},
  {b1, 0, d - 1}, {b2, 0, d - 1}, {c1, 0, d - 1}, {c2, 0, d - 1}]
BSumR[i1_, i2_, j1_, j2_, k1_, k2_, m1_, m2_, n1_, n2_, r1_, r2_, d_] :=
 Sum[B[j1, j2, i1, i2, c1, c2, a1, a2, d] B[k1, k2, c1, c2, m1, m2, b1, b2, d]
   B[b1, b2, a1, a2, n1, n2, r1, r2, d], {a1, 0, d - 1}, {a2, 0, d - 1},
  {b1, 0, d - 1}, {b2, 0, d - 1}, {c1, 0, d - 1}, {c2, 0, d - 1}]
```

Test - Modul. Tests n random combinations in dimension d
Output:

-Combination of indices producing non-zero result
-Left/right result
-Difference between left/right result

-Number of non-zero results
-Number of zero results
-Number of combinations satisfying braid equation

```
TestRun[d_, n_] :=
 Module[{dM = d, nM = n, vars, j, bsuml, bsumr, nonzeros = 0, zeros = 0, equal = 0, diff},
  For[i = 0, i < n, i = i + 1,
   vars = Append[Table[Random[Integer, d - 1], {j, 1, 12}], dM];
   bsuml = BSumL @@ vars;
   bsumr = BSumR @@ vars;
   diff = Simplify[bsuml - bsumr];
   If[bsuml ≠ 0 || bsumr ≠ 0,
    Print["Indizes: ", vars, "   Links:  ",
     bsuml, "   Rechts: ", bsumr , "   Differenz: ", diff];
    nonzeros++;
    If[diff == 0,
     equal++];
   ];
  ];
  Print["Zero: ", n - nonzeros];
  Print["Non-zero:     ", nonzeros];
  Print["Non-zero and equal: ", equal];
 ]
```

```
TestRun[4, 1000] // Timing
```

Indizes: {0, 1, 3, 0, 2, 3, 3, 2, 0, 1, 2, 1, 4}   Links: -8 Rechts: -4 Differenz: -4

Indizes: {0, 0, 0, 2, 0, 1, 0, 0, 0, 0, 0, 3, 4}   Links: 4 Rechts: -4 Differenz: 8

Indizes: {1, 3, 2, 2, 2, 2, 0, 2, 3, 3, 2, 2, 4}   Links: $-4\,i$ Rechts: $-8\,i$ Differenz: $4\,i$

Indizes: {1, 1, 2, 2, 2, 3, 1, 0, 1, 3, 3, 3, 4}   Links: $2\,i\,e^{\frac{i\pi}{4}}+2\,i\,e^{-\frac{3i\pi}{4}}+4\,e^{\frac{3i\pi}{4}}$ Rechts: $e^{-\frac{i\pi}{4}}-i\,e^{\frac{i\pi}{4}}-i\,e^{-\frac{3i\pi}{4}}+5\,e^{\frac{3i\pi}{4}}$ Differenz: 0

Indizes: {1, 2, 1, 2, 3, 1, 0, 0, 2, 1, 3, 0, 4}   Links: $2\,e^{-\frac{i\pi}{4}}-2\,i\,e^{\frac{i\pi}{4}}+2\,i\,e^{-\frac{3i\pi}{4}}+2\,e^{\frac{3i\pi}{4}}$ Rechts: $2\,e^{-\frac{i\pi}{4}}-2\,i\,e^{\frac{i\pi}{4}}-2\,i\,e^{-\frac{3i\pi}{4}}+2\,e^{\frac{3i\pi}{4}}$ Differenz: $-4\,(-1)^{3/4}$

Indizes: {3, 0, 3, 3, 2, 2, 1, 1, 2, 3, 1, 1, 4}   Links: $7\,e^{-\frac{i\pi}{4}}-2\,i\,e^{\frac{i\pi}{4}}+e^{\frac{3i\pi}{4}}$ Rechts: $5\,e^{-\frac{i\pi}{4}}-2\,i\,e^{\frac{i\pi}{4}}+2\,i\,e^{-\frac{3i\pi}{4}}+e^{\frac{3i\pi}{4}}$ Differenz: 0

Indizes: {1, 2, 1, 0, 0, 1, 1, 1, 1, 1, 0, 1, 4}   Links: 0 Rechts: $4\,i$ Differenz: $-4\,i$

Indizes: {2, 1, 0, 1, 2, 0, 1, 1, 1, 2, 2, 3, 4}   Links: $2\,e^{-\frac{i\pi}{4}}-2\,i\,e^{\frac{i\pi}{4}}+2\,i\,e^{-\frac{3i\pi}{4}}-2\,e^{\frac{3i\pi}{4}}$ Rechts: $-4\,i\,e^{-\frac{3i\pi}{4}}$ Differenz: $(6-6\,i)\sqrt{2}$

Indizes: {3, 3, 0, 1, 1, 1, 2, 2, 0, 2, 2, 1, 4}   Links: 4 Rechts: 8 Differenz: -4

Indizes: {2, 1, 1, 1, 2, 0, 1, 3, 2, 0, 2, 3, 4}   Links: $8\,i$ Rechts: 0 Differenz: $8\,i$

Indizes: {1, 2, 1, 0, 3, 3, 1, 0, 3, 3, 1, 2, 4}   Links: $-16\,i$ Rechts: $-4\,i$ Differenz: $-12\,i$

Indizes: {1, 0, 2, 1, 1, 3, 1, 1, 2, 2, 1, 1, 4}   Links: $2\,i\,e^{-\frac{i\pi}{4}}+7\,e^{\frac{i\pi}{4}}+e^{-\frac{3i\pi}{4}}-4\,i\,e^{\frac{3i\pi}{4}}$ Rechts: $-i\,e^{-\frac{i\pi}{4}}+3\,e^{\frac{i\pi}{4}}-e^{-\frac{3i\pi}{4}}-5\,i\,e^{\frac{3i\pi}{4}}$ Differenz: $4\,(-1)^{1/4}$

Indizes: {2, 3, 0, 0, 1, 2, 0, 1, 2, 3, 1, 1, 4}   Links: $-3\,i\,e^{-\frac{i\pi}{4}}-3\,e^{\frac{i\pi}{4}}+3\,e^{-\frac{3i\pi}{4}}-i\,e^{\frac{3i\pi}{4}}$ Rechts: $-5\,i\,e^{-\frac{i\pi}{4}}+2\,e^{-\frac{3i\pi}{4}}+i\,e^{\frac{3i\pi}{4}}$ Differenz: 0

Indizes: {3, 1, 0, 1, 1, 1, 0, 2, 3, 0, 1, 1, 4}   Links: $4\,e^{-\frac{i\pi}{4}}-2\,i\,e^{\frac{i\pi}{4}}+2\,i\,e^{-\frac{3i\pi}{4}}$ Rechts: $3\,e^{-\frac{i\pi}{4}}+2\,i\,e^{-\frac{3i\pi}{4}}+e^{\frac{3i\pi}{4}}$ Differenz: $-4\,(-1)^{3/4}$

Indizes: {0, 0, 3, 3, 2, 2, 1, 3, 0, 0, 0, 2, 4}   Links: $8\,i$ Rechts: $-8\,i$ Differenz: $16\,i$

Indizes: {0, 2, 3, 2, 2, 1, 2, 2, 3, 1, 0, 2, 4} Links: $-3 e^{-\frac{i\pi}{4}} + 3 i e^{\frac{i\pi}{4}} - 5 i e^{-\frac{3i\pi}{4}} + 5 e^{\frac{3i\pi}{4}}$ Rechts: $-e^{-\frac{i\pi}{4}} + 2 i e^{\frac{i\pi}{4}} - 2 i e^{-\frac{3i\pi}{4}} + 7 e^{\frac{3i\pi}{4}}$ Differenz: $4 (-1)^{3/4}$

Indizes: {2, 2, 3, 0, 2, 1, 1, 0, 1, 3, 1, 0, 4} Links: $i e^{\frac{i\pi}{4}} + i e^{-\frac{3i\pi}{4}} + 4 e^{\frac{3i\pi}{4}}$ Rechts: $4 e^{-\frac{i\pi}{4}} - i e^{\frac{i\pi}{4}} + i e^{-\frac{3i\pi}{4}} + 2 e^{\frac{3i\pi}{4}}$ Differenz: $8 (-1)^{3/4}$

Indizes: {3, 3, 1, 0, 1, 0, 1, 2, 1, 1, 3, 0, 4} Links: -4 Rechts: 8 Differenz: -12

Indizes: {0, 2, 1, 2, 2, 1, 1, 2, 3, 3, 3, 0, 4} Links: $e^{\frac{i\pi}{4}} + e^{-\frac{3i\pi}{4}}$ Rechts: $-i e^{-\frac{i\pi}{4}} + 4 e^{\frac{i\pi}{4}} + 4 e^{-\frac{3i\pi}{4}} + 3 i e^{\frac{3i\pi}{4}}$ Differenz: $4 (-1)^{1/4}$

Indizes: {0, 2, 2, 0, 3, 0, 3, 2, 0, 3, 2, 1, 4} Links: 8 Rechts: -8 Differenz: 16

Indizes: {0, 1, 0, 1, 2, 0, 2, 2, 0, 0, 0, 0, 4} Links: -8 Rechts: -8 Differenz: 0

Indizes: {2, 3, 3, 2, 2, 1, 1, 2, 0, 0, 2, 0, 4} Links: 0 Rechts: $4 i$ Differenz: $-4 i$

Indizes: {2, 0, 1, 2, 2, 2, 0, 3, 2, 0, 3, 1, 4} Links: $7 e^{\frac{i\pi}{4}} - e^{-\frac{3i\pi}{4}} - 4 i e^{\frac{3i\pi}{4}}$ Rechts: $-i e^{-\frac{i\pi}{4}} + 6 e^{\frac{i\pi}{4}} - 4 e^{-\frac{3i\pi}{4}} + i e^{\frac{3i\pi}{4}}$ Differenz: $4 (-1)^{1/4}$

Indizes: {2, 0, 3, 1, 3, 1, 1, 2, 0, 1, 3, 3, 4} Links: $-4 i e^{\frac{i\pi}{4}}$ Rechts: $5 e^{-\frac{i\pi}{4}} + 2 i e^{-\frac{3i\pi}{4}} + 3 e^{\frac{3i\pi}{4}}$ Differenz: 0

Indizes: {0, 2, 0, 3, 1, 3, 3, 0, 0, 1, 2, 3, 4} Links: $i e^{-\frac{i\pi}{4}} + 4 e^{-\frac{3i\pi}{4}} + i e^{\frac{3i\pi}{4}}$ Rechts: $-3 i e^{-\frac{i\pi}{4}} + e^{\frac{i\pi}{4}} - e^{-\frac{3i\pi}{4}} + 3 i e^{\frac{3i\pi}{4}}$ Differenz: 0

Indizes: {2, 2, 3, 1, 0, 1, 1, 1, 2, 1, 2, 2, 4} Links: $16 i$ Rechts: $12 i$ Differenz: $4 i$

Indizes: {0, 1, 3, 0, 0, 0, 1, 3, 0, 2, 2, 0, 4} Links: $-3 i e^{-\frac{i\pi}{4}} + 4 e^{-\frac{3i\pi}{4}} + i e^{\frac{3i\pi}{4}}$ Rechts: $2 i e^{-\frac{i\pi}{4}} - 4 e^{\frac{i\pi}{4}} + 2 i e^{\frac{3i\pi}{4}}$ Differenz: $-4 (-1)^{1/4}$

Indizes: {0, 1, 0, 3, 2, 3, 3, 2, 0, 1, 3, 0, 4} Links: 0 Rechts: $4 i$ Differenz: $-4 i$

Indizes: {1, 2, 2, 2, 3, 0, 1, 0, 0, 1, 1, 3, 4} Links: $2 i e^{-\frac{i\pi}{4}} - e^{\frac{i\pi}{4}} + 3 e^{-\frac{3i\pi}{4}} - 2 i e^{\frac{3i\pi}{4}}$ Rechts: $-4 i e^{-\frac{i\pi}{4}} - 2 e^{\frac{i\pi}{4}} + 2 e^{-\frac{3i\pi}{4}}$ Differenz: $8 (-1)^{1/4}$

Indizes: {3, 0, 3, 3, 0, 3, 0, 3, 1, 0, 1, 3, 4} Links: -4 Rechts: -16 Differenz: 12

Indizes: {0, 0, 0, 3, 3, 3, 1, 2, 3, 2, 3, 2, 4} Links: $-2 e^{-\frac{i\pi}{4}} + 2 i e^{-\frac{3i\pi}{4}} + 4 e^{\frac{3i\pi}{4}}$ Rechts: $-e^{-\frac{i\pi}{4}} - 3 i e^{\frac{i\pi}{4}} - i e^{-\frac{3i\pi}{4}} + e^{\frac{3i\pi}{4}}$ Differenz: $4 (-1)^{3/4}$

Indizes: {0, 1, 0, 3, 2, 1, 1, 1, 2, 2, 3, 2, 4} Links: $-i e^{-\frac{i\pi}{4}} - e^{\frac{i\pi}{4}} - 5 e^{-\frac{3i\pi}{4}} - i e^{\frac{3i\pi}{4}}$ Rechts: $-i e^{-\frac{i\pi}{4}} + e^{\frac{i\pi}{4}} - 5 e^{-\frac{3i\pi}{4}} - 3 i e^{\frac{3i\pi}{4}}$ Differenz: $-4 (-1)^{1/4}$

Indizes: {0, 3, 0, 3, 2, 2, 1, 2, 2, 1, 3, 1, 4} Links: $2 e^{-\frac{i\pi}{4}} - i e^{\frac{i\pi}{4}} - i e^{-\frac{3i\pi}{4}} + 2 e^{\frac{3i\pi}{4}}$ Rechts: $-2 e^{-\frac{i\pi}{4}} + 2 i e^{\frac{i\pi}{4}} - 4 i e^{-\frac{3i\pi}{4}}$ Differenz: $-8 (-1)^{3/4}$

Indizes: {2, 2, 0, 1, 2, 3, 2, 1, 1, 0, 1, 1, 4} Links: $-3 e^{-\frac{i\pi}{4}} - i e^{\frac{i\pi}{4}} - i e^{-\frac{3i\pi}{4}} - 3 e^{\frac{3i\pi}{4}}$ Rechts: $-4 i e^{-\frac{3i\pi}{4}} + 4 e^{\frac{3i\pi}{4}}$ Differenz: $-8 (-1)^{3/4}$

Indizes: {0, 1, 2, 0, 0, 0, 1, 2, 1, 3, 0, 0, 4} Links: $4 e^{-\frac{i\pi}{4}} - 2 i e^{\frac{i\pi}{4}} - 2 e^{\frac{3i\pi}{4}}$ Rechts: $3 e^{-\frac{i\pi}{4}} + 2 i e^{-\frac{3i\pi}{4}} + e^{\frac{3i\pi}{4}}$ Differenz: $-4 (-1)^{3/4}$

Indizes: {3, 2, 2, 2, 1, 0, 0, 1, 1, 1, 1, 2, 4} Links: $-4 i e^{-\frac{i\pi}{4}} - 4 e^{\frac{i\pi}{4}}$ Rechts: $-i e^{-\frac{i\pi}{4}} + 2 e^{\frac{i\pi}{4}} + 2 e^{-\frac{3i\pi}{4}} - i e^{\frac{3i\pi}{4}}$ Differenz: $-8 (-1)^{1/4}$

Indizes: {2, 3, 2, 3, 3, 2, 1, 1, 0, 2, 2, 1, 4} Links: $e^{-\frac{i\pi}{4}} - 3 i e^{\frac{i\pi}{4}} + i e^{-\frac{3i\pi}{4}} - 3 e^{\frac{3i\pi}{4}}$ Rechts: $e^{-\frac{i\pi}{4}} + i e^{\frac{i\pi}{4}} - i e^{-\frac{3i\pi}{4}} - e^{\frac{3i\pi}{4}}$ Differenz: $-8 (-1)^{3/4}$

Indizes: {3, 0, 0, 3, 1, 1, 3, 2, 3, 1, 2, 1, 4} Links: 0 Rechts: $-8 i$ Differenz: $8 i$

Indizes: {1, 2, 1, 0, 2, 2, 2, 3, 0, 3, 2, 2, 4} Links: -8 Rechts: -8 Differenz: 0

Indizes: {3, 2, 0, 1, 1, 1, 3, 3, 0, 1, 1, 0, 4} Links: $4 i$ Rechts: $4 i$ Differenz: 0

Indizes: {0, 0, 1, 2, 0, 0, 3, 0, 0, 1, 2, 1, 4}   Links: -4 Rechts: -8 Differenz: 4

Indizes: {3, 2, 2, 1, 2, 1, 1, 2, 1, 3, 1, 3, 4}   Links: 4   Rechts: 0   Differenz: 4

Indizes: {3, 1, 0, 0, 2, 1, 3, 1, 3, 2, 3, 3, 4}   Links:
$-2\, i\, e^{-\frac{i\pi}{4}} - 4\, e^{\frac{i\pi}{4}} + 2\, i\, e^{\frac{3 i\pi}{4}}$ Rechts: $3\, e^{\frac{i\pi}{4}} + 3\, e^{-\frac{3 i\pi}{4}}$ Differenz: $-8\,(-1)^{1/4}$

Indizes: {3, 1, 2, 1, 2, 3, 1, 3, 0, 1, 2, 1, 4}   Links: $8\, i$ Rechts: $4\, i$ Differenz: $4\, i$

Indizes: {0, 2, 3, 1, 2, 2, 2, 3, 3, 1, 0, 1, 4}   Links: 0   Rechts: 12   Differenz: -12

Indizes: {3, 2, 3, 1, 2, 1, 1, 1, 1, 3, 2, 0, 4}   Links: $-4\, i\, e^{\frac{i\pi}{4}} + 2\, i\, e^{-\frac{3 i\pi}{4}} - 2\, e^{\frac{3 i\pi}{4}}$
Rechts: $-3\, e^{-\frac{i\pi}{4}} + 2\, i\, e^{\frac{i\pi}{4}} - 2\, i\, e^{-\frac{3 i\pi}{4}} + 5\, e^{\frac{3 i\pi}{4}}$ Differenz: $-20\,(-1)^{3/4}$

Indizes: {2, 0, 0, 0, 0, 1, 0, 1, 2, 2, 0, 2, 4}   Links: -4 Rechts: -16 Differenz: 12

Indizes: {2, 0, 0, 0, 0, 2, 3, 1, 0, 3, 3, 2, 4}   Links:
$2\, e^{-\frac{i\pi}{4}} + 6\, i\, e^{\frac{i\pi}{4}}$ Rechts: $e^{-\frac{i\pi}{4}} - i\, e^{\frac{i\pi}{4}} - i\, e^{-\frac{3 i\pi}{4}} + e^{\frac{3 i\pi}{4}}$ Differenz: $4\,(-1)^{3/4}$

Indizes: {3, 1, 2, 1, 1, 0, 0, 2, 3, 3, 3, 1, 4}   Links:
$3\, i\, e^{-\frac{i\pi}{4}} + 5\, e^{\frac{i\pi}{4}} - 3\, e^{-\frac{3 i\pi}{4}} - 5\, i\, e^{\frac{3 i\pi}{4}}$ Rechts: $2\, e^{\frac{i\pi}{4}} + 2\, i\, e^{\frac{3 i\pi}{4}}$ Differenz: $(8 + 8\, i)\sqrt{2}$

Indizes: {0, 1, 3, 1, 3, 3, 2, 1, 1, 1, 3, 3, 4}   Links: 0   Rechts: -4 Differenz: 4

Indizes: {0, 1, 1, 2, 2, 0, 1, 2, 1, 1, 1, 0, 4}   Links: $-i\, e^{-\frac{i\pi}{4}} + 8\, e^{-\frac{3 i\pi}{4}} + 3\, i\, e^{\frac{3 i\pi}{4}}$
Rechts: $-i\, e^{-\frac{i\pi}{4}} + 2\, e^{\frac{i\pi}{4}} + 6\, e^{-\frac{3 i\pi}{4}} + 3\, i\, e^{\frac{3 i\pi}{4}}$ Differenz: $-4\,(-1)^{1/4}$

Indizes: {2, 2, 1, 1, 1, 3, 0, 3, 2, 0, 2, 3, 4}   Links: 8   Rechts: 12   Differenz: -4

Indizes: {2, 1, 3, 3, 3, 2, 3, 2, 0, 1, 1, 3, 4}   Links: $-8\, i$ Rechts: $16\, i$ Differenz: $-24\, i$

Indizes: {1, 3, 3, 1, 2, 2, 1, 1, 0, 2, 1, 3, 4}   Links: $-16\, i$ Rechts: $-4\, i$ Differenz: $-12\, i$

Indizes: {1, 0, 3, 2, 1, 1, 3, 1, 3, 3, 3, 3, 4}   Links: 4   Rechts: 8   Differenz: -4

Indizes: {0, 3, 0, 3, 1, 3, 0, 1, 3, 0, 2, 0, 4}   Links:
$3\, e^{\frac{i\pi}{4}} + 3\, e^{-\frac{3 i\pi}{4}}$ Rechts: $-3\, i\, e^{-\frac{i\pi}{4}} - e^{\frac{i\pi}{4}} + e^{-\frac{3 i\pi}{4}} + 3\, i\, e^{\frac{3 i\pi}{4}}$ Differenz: $8\,(-1)^{1/4}$

Indizes: {1, 0, 2, 3, 2, 2, 2, 3, 3, 1, 0, 1, 4}   Links: $i\, e^{-\frac{i\pi}{4}} + 4\, e^{\frac{i\pi}{4}} + i\, e^{\frac{3 i\pi}{4}}$
Rechts: $i\, e^{-\frac{i\pi}{4}} + 4\, e^{\frac{i\pi}{4}} - 4\, e^{-\frac{3 i\pi}{4}} - 3\, i\, e^{\frac{3 i\pi}{4}}$ Differenz: $-8\,(-1)^{1/4}$

Indizes: {2, 2, 2, 1, 1, 1, 1, 2, 3, 0, 1, 2, 4}   Links:
$2\, e^{-\frac{i\pi}{4}} - 2\, i\, e^{\frac{i\pi}{4}} - 4\, e^{\frac{3 i\pi}{4}}$ Rechts: $-e^{-\frac{i\pi}{4}} - i\, e^{\frac{i\pi}{4}} + 5\, i\, e^{-\frac{3 i\pi}{4}} - 3\, e^{\frac{3 i\pi}{4}}$ Differenz: 0

Zero: 942

Non-zero: 58

Non-zero and equal: 9

{1444.41, Null}

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Versicherung

Hiermit erkläre ich, die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben.

Kay-Michael Voit