Untersuchungen zur Integrabilität von Spinketten

Diplomarbeit

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April 2008

Korrigierte Fassung. Original eingereicht am 23. April 2008.

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Contents

List of Figures

1. Introduction

The work on integrable spin systems started about 70 years ago, when Hans Bethe solved the XXX-Heisenberg chain by means of what is called the coordinate Bethe ansatz today. From then on this field of research, in the classical as well as in the quantum mechanical case, was gained a lot of attention from both, physicists and mathematicians.
The physical importance is due to the general character of spin systems. They can serve as models for large classes of important systems like for example magnetic molecules [1] or those related to quantum information processing in solid state systems [2]. Besides they are used in certain limits to construct field and string theoretical models [3, 4, 5, 6]. Within all these applications mainly big systems or even such of infinite size are considered. Often huge spin quantum numbers or spin vectors respectively are of special interest. In the quantum case the consequences are particularly clear: As its dimension is given by $(2s+1)^N$, the size of the Hilbert space becomes bigger and bigger with growing N and s so that quickly no numerical methods can be used anymore. Therefore integrable spin systems come into play and the importance to make available as many of them as possible becomes clear.
Apart from concrete physical systems, the phenomenon of integrability, especially in the quantum mechanical case, is interesting on its own right. As the first quantum mechanical systems which have shown to be integrable in the sense of analytical diagonalization of the Hamiltonian were spin chains, a lot of effort has been made into this direction. Studying the integrability of spin systems quickly turned out to be an extremely productive approach to integrability and dominates this field so far.
The last high motivation to work on integrable spin systems lies more or less between both of the aforementioned. Today numerical calculations play an important role in physics. Although the efficiency of computers increases constantly, it is important to construct more powerful algorithms to meet the enormous standards. These have to be tested and integrable models are an ideal field for that.
Although they are concerned with the same class of physical systems, the research in the different branches, especially in the first two ones, lie far away from each other. This is essentially a consequence of the amazing algebraic structure on which the theory of integrability is based and which clearly lies somewhere between pure mathematics and mathematical physics, far away from the conventional theoretical branch. This thesis wants to close this gap by formulating the algebraic background in a compact way, choosing significant integrable spin systems and deriving them from the abstract algebraic theory.
So far no difference between classical and quantum mechanical systems has been made. Within this thesis they are treated in the same mathematical framework, essentially consisting of two formalisms. The first one is due to Ballesteros and Ragnisco and requires a so-called bialgebra with a non-empty center [7]. It allows

the construction of huge classes of classical and quantum mechanical integrable systems. By a simple extension of the theory to what will be called pseudo-coproducts from now on, it even covers the so-called $\mathcal{B}$-partitioned spin systems, which are due to Schmidt and Steinigeweg [8] and will be investigated more detailed. As far as the author knows, this extension is new.
The other formalism is based on quite an unsual approach of Drinfel´d which completely avoids the well-known Yang-Baxter equation [9]. It enables to derive integrable quantum systems just by considering quasicocommutative bialgebras. Quasicocommutativity corresponds to a postulate on the coproduct and the approach is therefore analogous to the one of Ballesteros and Ragnisco. As a concrete example the XXX-Heisenberg chain is solved by means of a Bethe ansatz resulting from this non-Yang-Baxter equation approach. This requires a somewhat "exotic" realization of the Yangian, which is a so-called Quantum Group constructed by Drinfel´d. As no physical systems have been worked out on this way so far, this "spin realization" should be due to the author.
In addition to the algebraic solution the aforementioned coordinate Bethe ansatz is considered. Both of the solutions are compared with each other and it is pointed out, in which cases there is an explicite transformation from the results of the first one, to the results of the second one. Although the relations are more or less obvious, they should be written down for the first time. A concrete calculation finishes the bridge to well-known methods in theoretical physics.
The whole thesis is devided into three sections. The first one is completely devoted to the mathematical background and does not contain any physical applications. It is used to adopt the notion of a bialgebra and the two additional structures mentioned above. The Yangian as the most important quasicocommutative bialgebra for this thesis is introduced and the "exotic" spin realization is presented.
It should be mentioned that the whole section is restricted to the most important mathematical concepts and completely avoids any further context like the deformation theoretical background and more general considerations concerning Quantum Groups. It is also important to point out that the mathematical introduction is worked out especially for the later applications and therefore differs in some points from the usual introductions in the standard references.
The second section clarifies the structure of integrability and the point of view from which it is tackled currently and hence by the two different approaches. These are presented in full detail and the significance of Quantum Groups in the context of integrability is stressed again.
In the third section the aforementioned systems will be constructed from the algebraic theory and special examples are considered to clarify the rather abstract background.
The thesis ends with a summary and a concise outlook on further questions.

2. The mathematical framework of integrability

The most important mathematical concept for the later considerations is the so-called bialgebra which combines the algebra and the coalgebra structure. The product of an algebra, as far as it causes no confusion denoted just by juxtaposition, is a linear map from the two-fold tensorial product of a vector space on the vector space itself

$$m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \tag{1}$$

postulated to be associative throughout this paper. The coalgebra structure now results by "reversing the arrow":

DEFINITION 2-1. *Let $\mathcal{A}$ be a vector space and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ a linear map. If it satisfies the so-called coassociativity condition*

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta, \tag{2}$$

it is called a coproduct and $(\mathcal{A}, \Delta)$ a coalgebra.

Every element of the tensor product of an algebra can be written as a linear superposition of tensor products of algebra elements. If the coefficients are included in the algebra elements, this gives an expression for the coproduct, usually referred to as Sweedler´s notation:

$$\Delta(X) = \sum_i X_{1i} \otimes X_{2i} \tag{3}$$

If τ denotes the switch operator

$$\tau\,(X \otimes Y) = Y \otimes X, \tag{4}$$

it is clear that

$$\underbrace{(m \otimes m) \circ (id \otimes \tau \otimes \mathrm{id})}_{:=m^{\otimes}}((X_1 \otimes Y_1) \otimes (X_2 \otimes Y_2)) = X_1 X_2 \otimes Y_1 Y_2 \tag{5}$$

makes the two-fold tensorial product of an algebra into an algebra itself. The coalgebra case is analogous:

LEMMA 2-1. *Let $(\mathcal{A}, \Delta)$ be a coalgebra. The two-fold tensorial product of $\mathcal{A}$ is made into a coalgebra by:*

$$\Delta^{\otimes} = (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta) \tag{6}$$

Proof. The proof results immediately by using Sweedler´s notation. By the assumption of coassociativity

$$\sum_i \Delta(X_{1i}) \otimes X_{2i} = \sum_i X_{1i} \otimes \Delta(X_{2i})$$
$$\sum_j \Delta(Y_{1i}) \otimes Y_{2i} = \sum_j Y_{1i} \otimes \Delta(Y_{2i})$$

and therefore [10]

$$\begin{aligned}
\left(\left(\Delta^{\otimes} \otimes \mathrm{id}^{(2)}\right) \circ \Delta^{\otimes}\right)(X \otimes Y) &= \tau_{23} \sum_{ij} \left(\Delta(X_{1i}) \otimes \Delta(Y_{1j})\right) \otimes X_{2i} \otimes Y_{2j} \\
&= \tau_{23}\tau_{45}\tau_{34} \sum_{ij} \left(\Delta(X_{1i}) \otimes X_{2i}\right) \otimes \left(\Delta(Y_{1j}) \otimes Y_{2j}\right) \\
&= \tau_{23}\tau_{45}\tau_{34} \sum_{ij} \left(X_{1i} \otimes \Delta(X_{2i})\right) \otimes \left(Y_{1j} \otimes \Delta(Y_{2j})\right) \\
&= \tau_{23}\tau_{45}\tau_{34}\tau_{34}\tau_{23} \sum_{ij} \left(X_{1i} \otimes Y_{1j}\right) \otimes \left(\Delta(X_{2i}) \otimes \Delta(Y_{2j})\right) \\
&= \tau_{45} \sum_{ij} \left(X_{1i} \otimes Y_{1j}\right) \otimes \left(\Delta(X_{2i}) \otimes \Delta(Y_{2j})\right) = \left(\left(\mathrm{id}^{(2)} \otimes \Delta^{\otimes}\right) \circ \Delta^{\otimes}\right)(X \otimes Y),
\end{aligned}$$

where $\mathrm{id}^{(2)}$ is the two-fold tensorial product of the identity map and τ_{ij} denotes the switch between the i-th and j-th position. Concerning the indices it should be reminded that the calculation takes place on the six-fold tensorial product of the algebra.

□

Now the bialgebra can be defined:

DEFINITION 2-2. *Let* $(\mathcal{A}, m)$ *be an algebra and* $(\mathcal{A}, \Delta)$ *a coalgebra. If the coproduct* Δ *is an algebra homomorphism, i.e., if*

$$\Delta(XY) = m^{\otimes}\left(\Delta(X) \otimes \Delta(Y)\right), \tag{7}$$

$(\mathcal{A}, m, \Delta)$ *is called a bialgebra.*

For later convenience from now on every algebra is assumed to have a unit element.
In the following it will be crucial to find concrete examples of abstract bialgebras. Hence the notion of a "representation" and a "realization" in the context of bialgebras has to be clarified:

DEFINITION 2-3. *Let $(\mathcal{A}, m, \Delta)$ and $(\mathcal{A}_1, m_1, \Delta_1)$ be two bialgebras. If $\rho : \mathcal{A} \to \mathcal{A}_1$ is an algebra homomorphism, ρ will be called a realization of $\mathcal{A}$. If V a is vector space and $\mathcal{A}_1 = \mathrm{End}(V)$, ρ will be called a representation and V the associated module.* [1]

It might be surprising that the generalization of the term "representation" to bialgebras only refers to the algebra structure. It will become clear later on that postulating ρ to be a coalgebra homomorphism

$$(\rho \otimes \rho)\,(\Delta(X)) = \Delta_1\,(\rho(X))\,, \tag{8}$$

where $X \in \mathcal{A}$, would be much to restrictive. Nevertheless, the coproduct plays an important role in the given context, as it combines two realizations $\rho_1 : \mathcal{A} \to \mathcal{A}_1$ and $\rho_2 : \mathcal{A} \to \mathcal{A}_2$ to a tensor realization:

$$(\rho_1 \otimes \rho_2) \circ \Delta : \mathcal{A} \to \mathcal{A}_1 \otimes \mathcal{A}_2 \tag{9}$$

If both of the realizations are representations associated with modules V_1 and V_2, this means in particular that $\mathcal{A}$ acts on $V_1 \otimes V_2$ by means of the tensor representation (9). If one of the most important coproduct structures (the "standard coproduct") for the subsequent applications is considered

$$\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i, \tag{10}$$

this action coincides with the one usually presented in the textbooks.
An important property of a bialgebra is its selfduality:

PROPOSITION 2-1. *Let $(\mathcal{A}, m, \Delta)$ be a bialgebra and $\mathcal{A}^*$ its dual space. The bialgebra structure on $\mathcal{A}$ induces a bialgebra structure on $\mathcal{A}^*$. If $\langle .|. \rangle : \mathcal{A}^* \times \mathcal{A} \to \mathbb{C}$ denotes the standard non-degenerated bilinear form, the bialgebra structure is given by:*

$$\langle \bar{m}(\bar{X} \otimes \bar{Y}) | X \rangle = \langle \bar{X} \otimes \bar{Y} | \Delta(X) \rangle \tag{11}$$

$$\langle \bar{\Delta}(\bar{X}) | X \otimes Y \rangle = \langle \bar{X} | XY \rangle \tag{12}$$

$(\mathcal{A}^*, \bar{m}, \bar{\Delta})$ will be called the dual bialgebra.

The dual product will again be denoted by juxtaposition, as long it causes no confusion.

Proof. It has to be shown that (11) and (12) define an (co-)product. Afterwards the algebra homomorphism property (7) has to be proven. It is convenient

[1] For a justification see A-1 in the appendix.

to start with the algebra structure:

$$\begin{aligned}
\langle \bar{X}(\bar{Y}\bar{Z})|X\rangle &= \langle \bar{X}\otimes(\bar{Y}\bar{Z})|\Delta(X)\rangle \\
&= \langle \bar{X}\otimes\bar{Y}\otimes\bar{Z}|(\mathrm{id}\otimes\Delta)\left(\Delta(X)\right)\rangle \\
&= \langle \bar{X}\otimes\bar{Y}\otimes\bar{Z}|(\Delta\otimes\mathrm{id})\left(\Delta(X)\right)\rangle \\
&= \langle (\bar{X}\bar{Y})\otimes\bar{Z}|\Delta(X)\rangle \\
&= \langle (\bar{X}\bar{Y})\bar{Z}|X\rangle
\end{aligned}$$

Now the coassociativity condition can be shown:

$$\begin{aligned}
\langle(\bar{\Delta}\otimes\mathrm{id})(\bar{\Delta}(\bar{X}))|X\otimes Y\otimes Z\rangle &= \langle\bar{\Delta}(\bar{X})|XY\otimes Z\rangle \\
&= \langle\bar{X}|(XY)Z\rangle \\
&= \langle\bar{X}|X(YZ)\rangle \\
&= \langle\bar{\Delta}(\bar{X})|X\otimes YZ\rangle \\
&= \langle(\mathrm{id}\otimes\bar{\Delta})(\bar{\Delta}(\bar{X}))|X\otimes Y\otimes Z\rangle
\end{aligned}$$

Finally the algebra homomorphism property can be considered:

$$\begin{aligned}
\langle\bar{\Delta}(\bar{X}\bar{Y})|X\otimes Y\rangle &= \langle\bar{X}\bar{Y}|XY\rangle \\
&= \langle\bar{X}\otimes\bar{Y}|\Delta(XY)\rangle \\
&= \langle\bar{X}\otimes\bar{Y}|m^{\otimes}(\Delta(X)\otimes\Delta(Y))\rangle \\
&= \langle(\mathrm{id}\otimes\tau\otimes\mathrm{id})\left(\bar{\Delta}(\bar{X})\otimes\bar{\Delta}(\bar{Y})\right)|\Delta(X)\otimes\Delta(Y)\rangle \\
&= \langle(\bar{m}\otimes\bar{m})\,(\mathrm{id}\otimes\tau\otimes\mathrm{id})\left(\bar{\Delta}(\bar{X})\otimes\bar{\Delta}(\bar{Y})\right)|X\otimes Y\rangle \\
&= \langle\bar{m}^{\otimes}\left(\bar{\Delta}(\bar{X})\otimes\bar{\Delta}(\bar{Y})\right)|X\otimes Y\rangle
\end{aligned}$$

□

To work on classical models, the notion of a Poisson bialgebra has to be defined:

DEFINITION 2-4. *Let* $(\mathcal{A}, m, \Delta)$ *be a bialgebra with a commutative product and a Poisson bracket* $\{.,.\}$. *If both of the structures are compatible, i.e., if* Δ *is a Poisson map*

$$\{\Delta(X), \Delta(Y)\} = \Delta\left(\{X, Y\}\right), \tag{13}$$

$(\mathcal{A}, m, \Delta, \{.,.\})$ *is called a Poisson bialgebra.*

A Poisson bracket is a Lie bracket which is a derivation. This means that it satisfies the Leibniz rule

$$L_Z(XY) = XL_Z(Y) + L_Z(X)Y, \tag{14}$$

where $L_Z(X) := \{X, Z\}$. The Poisson bracket on the left hand side of (13) is given by

$$\{X_1 \otimes Y_1, X_2 \otimes Y_2\}_{P\otimes Q} = \{X_1, X_2\}_P \otimes Y_1Y_2 + X_1X_2 \otimes \{Y_1, Y_2\}_Q \tag{15}$$

with $P = Q = \mathcal{A}$. If P and Q are multible tensor products of the same algebra, no subscript will be used, just as in Definition 2-4. Often the Poisson bracket is defined on the generators

$$\{X_i, X_j\} = \sum_k c_{ijk} X_k \tag{16}$$

and extended to the whole algebra as a derivation. The coefficients c_{ijk} are called the structure constants.
In the subsequent considerations two additional postulates on the respective (Poisson) bialgebras will be crucial. The first one claims a non-empty center $Z(\mathcal{A})$, defined by:

DEFINITION 2-5. *Let $(\mathcal{A}, m, \Delta, \{.,.\})$ be a (Poisson) bialgebra. An element $C \in \mathcal{A}$ lies in $Z(\mathcal{A}) \subset \mathcal{A}$, if and only if*

$$[C, X\} = 0 \tag{17}$$

for all $X \in \mathcal{A}$, where the simultanous notation $[.,.\}$ for the commutator and the Poisson bracket has been used.

In physics, central elements are usually called Casimir elements. The second important postulate will be the so-called quasicocommutativity:

DEFINITION 2-6. *A bialgebra $(\mathcal{A}, m, \Delta)$ is called quasicocommutative, if $\exists R \in \mathcal{A} \otimes \mathcal{A}$ for all $X \in \mathcal{A}$, so that the coproduct satisfies the following relation*

$$\Delta^{op}(X)R = R\Delta(X), \tag{18}$$

where $\Delta^{op} = \Delta \circ \tau$. The element R is called the universal R matrix, any representation $R^\rho := (\rho_1 \otimes \rho_2)(R)$ only R matrix.

The most important quasicocommutative bialgebra for the later applications is the so-called Yangian of the $\mathrm{sl}(2, \mathbb{C})$, denoted by $\mathrm{Y}(\mathrm{sl}(2, \mathbb{C}))$. It has to be mentioned that in the given context the $\mathrm{sl}(2, \mathbb{C})$ is not considered as the set of 2×2-matrices with complex entries and vanishing trace, but a three dimensional $\mathbb{C}$-vector space on which a certain Lie bracket is defined. This Lie bracket can be read off from the following Definition/Proposition 2-7 and the aforementioned set of matrices is understood as a special representation of this abtract $\mathrm{sl}(2, \mathbb{C})$.
Strictly speaking, the Yangian is "only" pseudoquasicocommutative, which will

be explained subsequently after the definition [11]:

DEFINITION/PROPOSITION 2-7. *The Yangian* $\mathrm{Y}(\mathrm{sl}(2,\mathbb{C}))$ *is an associative, algebra with a unit element and six generators* $e, f, g, J(e), J(f), J(h)$. *The first three ones constitute the usual universal enveloping algebra of the* $\mathrm{sl}(2,\mathbb{C})$ [2]*:*

$$\begin{aligned} [e,f] &= h \\ [h,e] &= 2e \\ [h,f] &= -2f \end{aligned} \tag{19}$$

Let $X, Y \in \{e, f, h\}$. *The rest of the generators can be written in terms of the respective* $\mathrm{sl}(2,\mathbb{C})$ *generators and have to satisfy the following relations:*

$$[X, J(Y)] = J([X,Y]) \tag{20}$$

$$[[J(e), J(f)], J(h)] = (J(e)f - eJ(f))\,h \tag{21}$$

The coproduct Δ *is defined on the generators by*

$$\begin{aligned} \Delta(X) &= X \otimes 1 + 1 \otimes X \\ \Delta(J(X)) &= J(X) \otimes 1 + 1 \otimes J(X) + \frac{1}{2}[X \otimes 1, t] \end{aligned} \tag{22}$$

and extended as an algebra homomorphism to the whole algebra. The element t *is defined by:*

$$t = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h \tag{23}$$

The Yangian is a so-called Quantum Group derived by Drinfel´d in the eighties [9]. Obviously a Quantum Group is not a group and the notion is ambigous. Either it refers to a special type of deformation of a Lie group´s function algebra or of its universal enveloping algebra, both of which are endowed with certain additional structures. The notion "deformation" essentially means that a new, parameter dependent structure is constructed so that it becomes the old one if the parameter vanishes. As the function- as well as the universal enveloping algebra contain the essential characteristics of the underlying group, the deformed ones can be associated with a new, a quantized group. The notion "quantized" mainly results from the fact that the deformed function algebra is non-commutative. Therefore both, the deformed function- and the deformed universal enveloping algebra are called Quantum Groups.
If $\mathrm{sl}(2,\mathbb{C})[u]$ denotes the set of polynomial maps from the complex numbers into

[2] For a definition see A-2 in the appendix.

sl$(2,\mathbb{C})$, the Yangian is the quantization of the associated universal enveloping algebra $\mathrm{U}_q\left(\mathrm{sl}(2,\mathbb{C})[u]\right)$ at $q = 1$. The proof that it is indeed a pseudoquasicocommutative bialgebra is beyond the scope of this thesis. It mainly requires the concrete quantization procedure, which is based on the deformation of $\mathrm{U}(\mathrm{sl}(2,\mathbb{C})[u])$ as a so-called Lie bialgebra. The associated additional structure is called the Lie bialgebra structure [12] and results from the following solution

$$r(u-v) = \frac{t}{u-v} \tag{24}$$

of the so-called Classical Yang-Baxter equation

$$\lceil r_{12}, r_{13}\rceil + \lceil r_{12}, r_{23}\rceil + \lceil r_{13}, r_{23}\rceil = 0, \tag{25}$$

where $\lceil .,.\rceil$ denotes the Lie-bracket and, adopting Sweedler´s notation, the quantities on the left hand side are given by:

$$\begin{aligned} r_{12} &= \sum_i r_{1i} \otimes r_{2i} \otimes 1 \\ r_{13} &= \sum_i r_{1i} \otimes 1 \otimes r_{2i} \\ r_{23} &= \sum_i 1 \otimes r_{1i} \otimes r_{2i} \end{aligned} \tag{26}$$

The bialgebra property now results from the deformation process, the pseudoquasicocommutativity by considering $r(u-v)$ within this process [13, 14]. As far as the author knows, a proof of the pseudoquasicocommutativity only based on the definition has not been given so far, whereas it can be shown that (21) ensures the algebra homomorphism property of the coproduct [15].
In the following two important realizations of the Yangian $\mathrm{Y}(\mathrm{sl}(2,\mathbb{C}))$ will be presented, both of which share the same basic idea. As the representation theory of the sl$(2,\mathbb{C})$ is well-known, it is no problem to find realizations for the first part of the generators. The much more difficult task is to construct the second part based on the first one. For the purpose of this paper, already the easiest imaginable method is completely convenient [13]:

PROPOSITION 2-2. *Let $X \in \{\sigma^i\}$, $i = +,-,3$ denote the usual creation, annihilation and third Pauli matrix. If $\lambda \in \mathbb{C}$ denotes an arbitrary complex number, the following identifications, together with the usual matrix product, give a representation of the Yangian $\mathrm{Y}(\mathrm{sl}(2,\mathbb{C}))$, which will be called the "spin representation":*

$$\{e, f, h\} \to \{\sigma^+, \sigma^-, \sigma^3\} \tag{27}$$

$$J(X) \to \lambda X \tag{28}$$

The parameter λ is called the "evaluation parameter" and the associated module will be denoted in terms of usual $\mathrm{sl}(2,\mathbb{C})$*-modules by* $V(1)_\lambda$ [3].

As the identity (20) results immediately and both sides of (21) are equal to zero, an explicit proof of the proposition is unnecessary. This representation is a special so-called "evaluation representation". Obviously it is no coalgebra homomorphism, which clarifies Definition 2-3. Subsequently the spin representation will be used to construct a second realization by modifying the product structure.
Before taking this important step, the R matrix associated with the spin representation will be constructed. As mentioned above, the Yangian is pseudoquasicocommutative. This means that its universal R matrix is not an element of $\mathrm{Y}(\mathrm{sl}(2,\mathbb{C}))\otimes\mathrm{Y}(\mathrm{sl}(2,\mathbb{C}))$, but of the two-fold tensorial product of the associated set of formal power series in two arbitrary parameters λ and μ, denoted by $\mathrm{Y}(\mathrm{sl}(2,\mathbb{C}))[[\lambda/\mu]]$. As

$$\mathrm{Y}(\mathrm{sl}(2,\mathbb{C}))[[\lambda]]\otimes\mathrm{Y}(\mathrm{sl}(2,\mathbb{C}))[[\mu]]\cong(\mathrm{Y}(\mathrm{sl}(2,\mathbb{C}))\otimes\mathrm{Y}(\mathrm{sl}(2,\mathbb{C})))[[\nu]], \qquad (29)$$

where ν is a function of λ and μ, it is clear that, despite of the pseudoquasicocommutativity, in principle the R matrix can be calculated by applying the two-fold tensorial product of a realization to the universal R matrix, giving a function $R^\rho = R^\rho(\nu)$. Unfortunately there is no explicit expression for it, which makes it necessary to construct it due to general considerations. Mainly following [13, 16, 17], we begin with the following observation:

LEMMA 2-2. *The two-fold tensorial product of the spin representation is reducible, if and only if the associated evaluation parameters λ and μ satisfy $\lambda-\mu-1=0$.*

Proof. The module associated with the tensor product of the spin representation is reducible as an $\mathrm{sl}(2,\mathbb{C})$ module:

$$V(1)_\lambda\otimes V(1)_\mu\cong V(2)\oplus V(0)$$

It will be shown that the second component is a Yangian module, if and only if $\lambda-\mu-1=0$.
"$\Rightarrow$": If

$$\{|1\rangle,|0\rangle\}$$

denote the usual spin up and down states, the second component is spanned by the highest weight vector [4]

$$|10\rangle-|01\rangle.$$

[3] $V(m)$ denotes the module associated with $S=m/2$.
[4] This means that it is annihilated under the action of the creation operator [17].

The action of the non-sl$(2,\mathbb{C})$ part of the Yangian on this vector has to be calculated. Due to (9), this is given by the representation of the coproduct. The commutator in the third term of (22) can be calculated using the commutation relations

$$\begin{aligned}
\frac{1}{2}\left[\sigma^{+}\otimes 1, t\right] &= \frac{1}{2}\left(\left[\sigma^{+}\otimes 1, \sigma^{+}\otimes\sigma^{-}\right]+\left[\sigma^{+}\otimes 1, \sigma^{-}\otimes\sigma^{+}\right]\right) \\
&+ \frac{1}{4}\left[\sigma^{+}\otimes 1, \sigma^{3}\otimes\sigma^{3}\right] \\
&= \frac{1}{2}\left(\left[\sigma^{+}, \sigma^{-}\right]\otimes\sigma^{+}+\frac{1}{2}\left[\sigma^{+}, \sigma^{3}\right]\otimes\sigma^{3}\right) \\
&= \frac{1}{2}\sigma^{3}\otimes\sigma^{+}-\frac{1}{2}\sigma^{+}\otimes\sigma^{3},
\end{aligned}$$

so that it follows

$$\begin{aligned}
J(\sigma^{+}).\left(|10\rangle-|01\rangle\right) &= \left(\lambda\sigma^{+}\otimes 1+\mu 1\otimes\sigma^{+}+\frac{1}{2}\left[\sigma^{+}\otimes 1, t\right]\right)\left(|10\rangle-|01\rangle\right) \\
&= (\lambda-\mu-1)\left(-|11\rangle\right),
\end{aligned}$$

where the dot denotes the action on the tensor module. The results for the other elements can be obtained with the same calculation:

$$\begin{aligned}
J(\sigma^{-}).\left(|10\rangle-|01\rangle\right) &= (\lambda-\mu-1)|00\rangle \\
J(\sigma^{3}).\left(|10\rangle-|01\rangle\right) &= (2\lambda-2\mu-1)\left(|10\rangle-|01\rangle\right)
\end{aligned}$$

It follows immediately that if $\lambda-\mu-1=0$, the component $V(0)$ is invariant under the Yangian action and hence the tensor product of the spin representation is reducible.
"$\Leftarrow$": If $V(0)$ is assumed to be a Yangian subrepresentation, it follows immediately from the above calculations that $\lambda-\mu-1=0$. Otherwise it would not be invariant under the Yangian action, contradicting the assumption.

□

Now the R matrix can be calculated:

PROPOSITION 2-3. *The R matrix of the spin representation associated with the evaluation parameters λ and μ is, up to a scalar multiple, given by the so-called Yang matrix*

$$R^{\rho}(\lambda,\mu)=R^{\rho}(\mu-\lambda)=\frac{i(\mu-\lambda)-i\tau}{i(\mu-\lambda)-i}, \tag{30}$$

where τ in this case has to be understood as the matrix resulting from the representation of the switch operator in the usual spin basis.

Before giving the proof of this proposition, it should be pointed out that obviously the evaluation parameter, considered as a variable, realizes the aforementioned spectral parameter. On the first sight this might be somewhat surprising, as the evaluation parameter is a part of the representation and not the underlying algebra. For an explanation it is helpful to consider the sketch (Figure 1). The extended Yangian can be considered to be composed of three subalgebras. The first one is given by power series in the spectral parameter λ with coefficients c_i which are exclusively generated by e, f, g. This is depicted as the first box. The second box symbolizes the subalgebra related to coefficients consisting of mixed products exclusively. The last subalgebra with coefficients only generated by $J(e), J(f), J(h)$ is depicted as the last box. Choosing a parameter independent representation ρ

$$\{e, f, g\} \xrightarrow{\rho} \left\{\sigma^+, \sigma^-, \sigma^3\right\}, \tag{31}$$

the image of the first subalgebra consists of power series with coefficients generated by the above σ^i. This however is nothing but the spin representation of the normal, non-extended Yangian. Considering (29), it is clear that the R

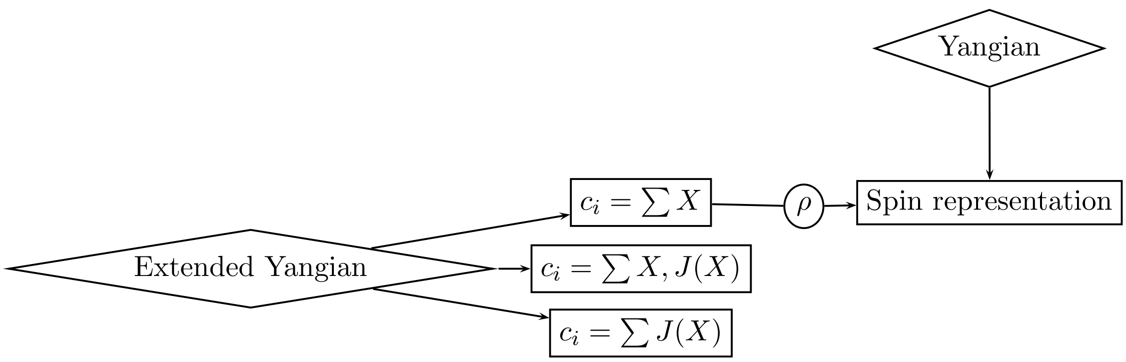

Figure 1: The relation between the spectral and the evaluation parameter

matrix results as given in (30). This argumentation only works for pure evaluation representations. If for example the Yangian is realized on a tensor product of evaluation representations, additional structures have to be considered. It is interesting in this context that, due to [16], every irreducible, finite dimensional representation of the Yangian is identical up to isomorphism to a tensor product of evaluation representations.
It should be mentioned that this unsual point of view means that in the later considerations parameter independent representations will be used.

Proof. The sl$(2, \mathbb{C})$ part of the Yangian coproduct is equal to its opposite in the sense of (18). Hence the R matrix can be considered as a self-intertwiner of the respective sl$(2, \mathbb{C})$ tensor module. Therefore, as a direct consequence of Schur´s

Lemma, it can be written as a sum of projectors on the irreducible components of the module. In the case of the spin representation

$$\begin{aligned} V(2) &= \left\langle |11\rangle, \frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right), |00\rangle \right\rangle \\ V(0) &= \left\langle \frac{1}{\sqrt{2}}\left(|10\rangle - |01\rangle\right) \right\rangle, \end{aligned}$$

where $\langle \ldots \rangle$ denotes the span. This means:

$$\begin{aligned} R^\rho &= c_1 \left(|11\rangle\langle 11| + \frac{1}{2}\left(|10\rangle + |01\rangle\right)\left(\langle 10| + \langle 01|\right) + |00\rangle\langle 00| \right) \\ &+ \frac{c_2}{2}\left(|10\rangle - |10\rangle\right)\left(\langle 10| - \langle 01|\right) \end{aligned} \quad (32)$$

To calculate the two coefficients a, in the sense of the Yangian action, slightly more general aspect has to be considered:
Let ρ_1 and ρ_2 denote two arbitrary representations of a quasicocommutative bialgebra associated with modules V_1 and V_2 and let I be defined by

$$I = \tau_{12} \circ (\rho_1 \otimes \rho_2)(R) : V_1 \otimes V_2 \to V_2 \otimes V_1.$$

It has to be reminded that the action of the algebra on the tensor product of two modules, again denoted by a simple dot in the following, is given by (9). A short calculation shows that I commutes with it:

$$\begin{aligned} IX &= \tau_{12} \circ (\rho_1 \otimes \rho_2)(R) \circ (\rho_1 \otimes \rho_2)(\Delta(X)) \\ &= \tau_{12} \circ (\rho_1 \otimes \rho_2)(R\Delta(X)) \\ &= \tau_{12} \circ (\rho_1 \otimes \rho_2)(\Delta^{op}(X)R) \\ &= \tau_{12} \circ (\rho_1 \otimes \rho_2)((\tau \circ \Delta)(X)R) \\ &= (\rho_2 \otimes \rho_1)(\Delta(X)) \circ \tau_{12} \circ (\rho_1 \otimes \rho_2)(R) = XI \end{aligned} \quad (33)$$

This means that if the tensor module is irreducible, I is unique up to scalar multible due to Schur´s Lemma.
Applying the relation (33) to the spin representation for $x = J(\sigma^+)$ gives an expression for the two coefficients and hence determines them uniquely if $\lambda - \mu - 1 \neq 0$, as shown in Lemma 2-2. The action on the tensor module has been calculated in the proof of Lemma 2-2 and it remains to calculate the action on the image of I. As this switches not only the states, but also the modules, the order of the parameters changes:

$$\begin{aligned} J(\sigma^+).\left(|01\rangle - |10\rangle\right) &= -\left(\mu\sigma^+ \otimes 1 + \lambda 1 \otimes \sigma^+ + \frac{1}{2}\left[\sigma^+ \otimes 1, t\right]\right)\left(|10\rangle - |01\rangle\right) \\ &= -\left(\mu - \lambda - 1\right)\left(-|11\rangle\right) \end{aligned}$$

Hence (33) can be used, giving:

$$\begin{aligned}(\lambda-\mu-1)\,c_0\,(-|11\rangle) &= (\lambda-\mu-1)\,I.\,(-|11\rangle)\\ &= I.J(\sigma^+).\,(|10\rangle-|01\rangle)\\ &= J(\sigma^+).I.\,(|10\rangle-|01\rangle)\\ &= c_1 J(\sigma^+).\,(|01\rangle-|10\rangle) = -c_1\,(\mu-\lambda-1)\,(-|11\rangle)\end{aligned}$$

This leads to the following postulate on the ratio of the both coefficients

$$\frac{c_1}{c_0} = \frac{\mu-\lambda+1}{\mu-\lambda-1} = \frac{i(\mu-\lambda)+i}{i(\mu-\lambda)-i}.$$

Setting $c_0 = 1$ and inserting the result of the above calculation gives

$$\begin{aligned}R^\rho(\mu-\lambda) &= P_0 + \frac{i(\mu-\lambda)+i}{i(\mu-\lambda)-i}P_1\\ &= \frac{1}{i(\mu-\lambda)-i}\left((i(\mu-\lambda)-i)\,P_0 + (i(\mu-\lambda)+i)\,P_1\right)\\ &= \frac{1}{i(\mu-\lambda)-i}\left(i(\mu-\lambda)+i\,(P_1-P_0)\right)\\ &= \frac{1}{i(\mu-\lambda)-i}\left(i(\mu-\lambda)-i\tau\right),\end{aligned}$$

where P_0 and P_1 denote the projectors on the respective irreducible $\mathrm{sl}(2,\mathbb{C})$ components. As this is exactly the Yang matrix, the proof is finished.

□

Now it is time to construct the second realization of the Yangian, crucial for the later applications. As mentioned above, this new realization will essentially result by modifying the product of the spin representation, just considering the abstract spin 1/2 operators with $\hbar = 1$ instead of the associated matrices in the S^3 eigenbasis.
Given an arbitrary realization of a bialgebra, the associated dual space contains exactly the same elements (at least in the cases relevant for this thesis), where in the given context the standard non-degenerated and symmetric bilinear form for real matrices will be considered:

$$\langle A|B\rangle = \mathrm{Tr}\,(AB) \tag{34}$$

The difference between the bialgebra and its dual lies in the product- and coproduct structure resulting from (11) and (12). The idea of the "modified spin representation" is now to impose a certain dual coproduct and using (12) to define the new product. The construction of the new realization is then completed by proving the defining relations of Definition/Proposition 2-7.

To be able to define the aforementioned dual coproduct, a new set of generators of the space assumed to be the dual of the new realization has to be chosen and arranged into what will be called the T matrix from now on [5]:

$$T(\lambda) = \begin{pmatrix} \lambda + S^3 & S^- \\ S^+ & \lambda - S^3 \end{pmatrix} \tag{35}$$

The dual coproduct associated with this T matrix will be considered as the following:

DEFINITION/PROPOSITION 2-4. *Let $t_{ij}(\lambda)$ denote the (i,j)-th entry of $T(\lambda)$. The map $\bar{\Delta}$, defined on the generators by*

$$\bar{\Delta}(t_{ij}(\lambda)) = \sum_{k=1}^{2} t_{ik}(\lambda) \otimes t_{kj}(\lambda) \tag{36}$$

and extended as an algebra homomorphism, defines a coproduct on the algebra generated by the elements of $T(\lambda)$.

Proof. The linearity is evident and the coassociativity results immediately due to general considerations, not involving $T(\lambda)$ explicitely

$$\begin{aligned}
\left(\mathrm{id} \otimes \bar{\Delta}\right)\left(\bar{\Delta}(t_{ij})\right) &= \sum_{k} t_{ik} \otimes \bar{\Delta}(t_{kj}) \\
&= \sum_{k,l} t_{ik} \otimes t_{kl} \otimes t_{lj} \\
&\overset{k\leftrightarrow l}{=} \sum_{k,l} t_{il} \otimes t_{lk} \otimes t_{kj} \\
&= \sum_{k} \bar{\Delta}(t_{ik}) \otimes t_{kj} = \left(\bar{\Delta} \otimes \mathrm{id}\right)\left(\bar{\Delta}(t_{ij})\right),
\end{aligned}$$

where the paramater λ has been omitted for clarity.

□

Now the defining relations can be checked to finish the construction of the second realization of the Yangian $\mathrm{Y}(\mathrm{sl}(2,\mathbb{C}))$. As it could not be found anywhere in the literature, it should be due to the author.

THEOREM 2-1. *Let $\lambda \in \mathbb{C}$ be an arbitrary complex number and $X \in \{e, f, h\}$. The spin operators and the product m, given by (36) via the selfduality relation (12), realize the Yangian $\mathrm{Y}(\mathrm{sl}(2,\mathbb{C}))$:*

$$\{e, f, h\} \to \left\{S^+, S^-, 2S^3\right\} \tag{37}$$

[5]Often the T matrix is called "Lax operator."

$$J(X) \to \lambda X \tag{38}$$

This realization will be called the "spin realization".

Proof. As the elements of $T(\lambda)$ are considered to be generators, any element $t(\lambda)$ of the associated algebra can be written as a sum of products of them

$$t = \sum_k a^{i_1 \ldots i_k} t_{i_1} \ldots t_{i_k},$$

where the two matrix indices have been replaced by just one new index, running from one to four, and the parameter λ is omitted for clarity. Concerning the indices i_l, Einstein´s summation rule is considered.
Now the relations of Definition/Proposition 2-7 have to be checked. As the dual elements are no algebra homomorphisms, this has to be done by complete induction over k, starting with $k = 1$. This means that every relation has to be tested for every generator. In the following it is convenient to write down the product explicitely:

$$\begin{aligned}\left[m\left(S^+ \otimes S^-\right) - m\left(S^- \otimes S^+\right)\right](t_{ij}) &= \langle m\left(S^+ \otimes S^-\right)|t_{ij}\rangle - \langle m\left(S^- \otimes S^+\right)|t_{ij}\rangle \\ &= \langle S^+ \otimes S^- | \bar{\Delta}\left(t_{ij}\right)\rangle - \langle S^- \otimes S^+ | \bar{\Delta}\left(t_{ij}\right)\rangle \\ &= \sum_{l=1}^{2}\left[\langle S^+|t_{il}\rangle\langle S^-|t_{lj}\rangle - \langle S^-|t_{il}\rangle\langle S^+|t_{lj}\rangle\right]\end{aligned}$$

Recognizing

$$\begin{aligned}\langle T(\lambda)|S^+\rangle &= \sigma^+ \\ \langle T(\lambda)|S^-\rangle &= \sigma^- \\ \langle T(\lambda)|2S^3\rangle &= \sigma^3,\end{aligned}$$

it follows:

$$\begin{aligned} i = j = 1 \to \quad & \ldots \quad = 1 = 2S^3\left(t_{11}\right) \\ i = 1, j = 2 \to \quad & \ldots \quad = 0 = 2S^3\left(t_{12}\right) \\ i = 2, j = 1 \to \quad & \ldots \quad = 0 = 2S^3\left(t_{21}\right) \\ i = j = 2 \to \quad & \ldots \quad = -1 = 2S^3\left(t_{22}\right)\end{aligned}$$

The remaining two indentities are resulting by the same calculation:

$$\left[m\left(2S^3 \otimes S^+\right) - m\left(S^+ \otimes 2S^3\right)\right](t_{ij}) = \sum_{l=1}^{2}\left[\langle 2S^3|t_{il}\rangle\langle S^+|t_{lj}\rangle - \langle S^+|t_{il}\rangle\langle 2S^3|t_{lj}\rangle\right]$$

$\Rightarrow$

$$\begin{aligned}
i = j = 1 \rightarrow \quad \ldots \quad &= 0 = 2S^+ (t_{11}) \\
i = 1, j = 2 \rightarrow \quad \ldots \quad &= 2 = 2S^+ (t_{12}) \\
i = 2, j = 1 \rightarrow \quad \ldots \quad &= 0 = 2S^+ (t_{21}) \\
i = j = 2 \rightarrow \quad \ldots \quad &= 0 = 2S^+ (t_{22})
\end{aligned}$$

$$\left[m\left(2S^3 \otimes S^-\right) - m\left(S^- \otimes 2S^3\right)\right](t_{ij}) = \sum_{l=1}^{2} \left[\langle 2S^3|t_{il}\rangle\langle S^-|t_{lj}\rangle - \langle S^-|t_{il}\rangle\langle 2S^3|t_{lj}\rangle\right]$$

$\Rightarrow$

$$\begin{aligned}
i = j = 1 \rightarrow \quad \ldots \quad &= 0 = 2S^- (t_{11}) \\
i = 1, j = 2 \rightarrow \quad \ldots \quad &= 0 = 2S^- (t_{12}) \\
i = 2, j = 1 \rightarrow \quad \ldots \quad &= -2 = 2S^- (t_{21}) \\
i = j = 2 \rightarrow \quad \ldots \quad &= 0 = 2S^- (t_{22})
\end{aligned}$$

Now the conclusion $k \rightarrow k+1$ can be made. For that it is convenient to change to the one index notation and to start with just one term of the first commutator:

$$\begin{aligned}
\langle m\left(S^+ \otimes S^-\right)|\bar{m}\left(t_1 \ldots t_k \otimes t_{k+1}\right)\rangle &= \langle \Delta\left(m\left(S^+ \otimes S^-\right)\right)|t_1 \ldots t_k \otimes t_{k+1}\rangle \\
&= \langle m^{\otimes}\left(\Delta(S^+) \otimes \Delta(S^-)\right)|t_1 \ldots t_k \otimes t_{k+1}\rangle \\
&= \langle m\left(S^+ \otimes S^-\right) \otimes 1 + S^- \otimes S^+|t_1 \ldots t_k \otimes t_{k+1}\rangle \\
&+ \langle S^+ \otimes S^- + 1 \otimes m\left(S^+ \otimes S^-\right)|t_1 \ldots t_k \otimes t_{k+1}\rangle \\
&= \langle m\left(S^+ \otimes S^-\right)|t_1 \ldots t_k\rangle\langle 1|t_{k+1}\rangle + \langle S^-|t_1 \ldots t_k\rangle\langle S^+|t_{k+1}\rangle \\
&\quad + \langle S^+|t_1 \ldots t_k\rangle\langle S^-|t_{k+1}\rangle + \langle 1|t_1 \ldots t_k\rangle\langle m\left(S^+ \otimes S^-\right)|t_{k+1}\rangle
\end{aligned}$$

With the same calculation for the remaining term it follows for the commutator:

$$\begin{aligned}
&\left[m\left(S^+ \otimes S^-\right) - m\left(S^- \otimes S^+\right)\right]\left(\bar{m}\left(t_1 \ldots t_k \otimes t_{k+1}\right)\right) = \\
&\qquad \langle m\left(S^+ \otimes S^-\right) - m\left(S^- \otimes S^+\right)|t_1 \ldots l_k\rangle\langle 1|t_{k+1}\rangle \\
&\qquad + \langle 1|t_1 \ldots t_k\rangle\langle m\left(S^+ \otimes S^-\right) - m\left(S^- \otimes S^+\right)|t_{k+1}\rangle \\
&\qquad + \langle S^-|t_1 \ldots t_k\rangle\langle S^+|t_{k+1}\rangle + \langle S^+|t_1 \ldots t_k\rangle\langle S^-|t_{k+1}\rangle \\
&\qquad - \langle S^+|t_1 \ldots t_k\rangle\langle S^-|t_{k+1}\rangle - \langle S^-|t_1 \ldots t_k\rangle\langle S^+|t_{k+1}\rangle \\
&\qquad = \langle m\left(S^+ \otimes S^-\right) - m\left(S^- \otimes S^+\right)|t_1 \ldots l_k\rangle\langle 1|t_{k+1}\rangle \\
&\qquad\qquad + \langle 1|t_1 \ldots t_k\rangle\langle m\left(S^+ \otimes S^-\right) - m\left(S^- \otimes S^+\right)|t_{k+1}\rangle
\end{aligned}$$

Now the conclusion of the induction can be finished, using the assumption:

$$
\begin{aligned}
&\left[m\left(S^{+}\otimes S^{-}\right)-m\left(S^{-}\otimes S^{+}\right)\right]\left(\bar{m}\left(t_1\ldots t_k\otimes t_{k+1}\right)\right)\\
&\qquad\qquad= \langle m\left(S^{+}\otimes S^{-}\right)-m\left(S^{-}\otimes S^{+}\right)|t_1\ldots l_k\rangle\langle 1|t_{k+1}\rangle\\
&\qquad\qquad+\langle 1|t_1\ldots t_k\rangle\langle m\left(S^{+}\otimes S^{-}\right)-m\left(S^{-}\otimes S^{+}\right)|t_{k+1}\rangle\\
&\qquad\qquad\qquad\qquad= \langle 2S^3|t_1\ldots t_k\rangle\langle 1|t_{k+1}\rangle+\langle 1|t_1\ldots t_k\rangle\langle 2S^3|t_{k+1}\rangle
\end{aligned}
$$

The last line is obviously the action of $2S^3$ on the above argument:

$$
\begin{aligned}
2S^3\left(\bar{m}\left(t_1\ldots l_k\otimes l_{k+1}\right)\right) &= \langle\Delta(2S^3)|t_1\ldots t_k\otimes t_{k+1}\rangle\\
&= \langle 2S^3\otimes 1+1\otimes 2S^3|t_1\ldots t_k\otimes t_{k+1}\rangle\\
&= \langle 2S^3|t_1\ldots t_k\rangle\langle 1|t_{k+1}\rangle+\langle 1|t_1\ldots t_k\rangle\langle 2S^3|t_{k+1}\rangle
\end{aligned}
$$

The remaining identities result just by replacing S^+ and S^- by the other respective generators. The conclusion to arbitrary t results immediately from the linearity of the dual elements. The relation (20) again is obvious and due to the above calculations both sides of (21) are equal to zero again. This completes the proof.

□

3. From bialgebras to integrability

In classical mechanics the phase space of a physical system with N degrees of freedom is given by a $2N$ dimensional Poisson manifold. This means that the function algebra is endowed with a Poisson structure. Due to the Liouville-Arnold theorem [18], a system is called "integrable" if there are N constants of motion f_i which are in involution with respect to the associated Poisson bracket

$$\{f_i, f_j\} = 0, \tag{39}$$

where i,j=1,2,...,N
Accordingly, in the classical framework, integrability has a precise meaning. In quantum mechanics this is not the case. Despite for huge effort, no quantum analogue of the Liouville-Arnold theorem has been deduced so far. The only approach to integrability in quantum mechanics is given by methods which enable to calculate analytical expressions for "relevant quantities" of a given physical system. Concerning the diagonalization of a Hamiltonian it is well-known from the angular momentum calculations in elementary quantum mechanics that this problem can be tackled by complete sets of mutually commuting operators. Completeness means that the eigenstates can be labelled uniquely by the eigenvalues. Using the selfadjointness and the commutation relations, the range of the labels and the coefficients associated with an arbitrary basis can be calculated, solving the problem completely.
Another way is given by the so-called Bethe ansatz. This aims at the parameter dependent construction of creation and annihilation operators relative to a reference state and was first used by Hans Bethe for solving the XXX-S=1/2-Heisenberg chain. Usually the Bethe ansatz appears with a certain specification, which indicates its origin. The calculation done by Bethe is called the coordinate Bethe ansatz, whereas the notion of algebraic Bethe ansatz refers to the modern procedure. This is based on operators, satisfying the so-called RTT relations

$$R(\lambda, \mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R(\lambda, \mu), \tag{40}$$

where the usual matrix product is considered and $R(\lambda, \mu)$ is a $n^2 \times n^2$ matrix with complex valued entries, varying with two so-called spectral parameters. If E_n denotes the $n \times n$ unit matrix and $T(\lambda)$ an $n \times n$ matrix of operators, $T_1(\lambda)$ and $T_2(\lambda)$ are defined by:

$$\begin{aligned} T_1(\lambda) &= T(\lambda) \otimes E_n \\ T_2(\lambda) &= E_n \otimes T(\lambda) \end{aligned} \tag{41}$$

Elements satisfying the RTT relations can be considered as generators of an RTT algebra. If the Hamiltonian H of a system is an element of an RTT algebra, the associated relations (40) give a basis to interpret certain elements as creation

or annihilation operators relative to a known eigenstate of H. This "algebraic ansatz" normally gives a set of so-called Bethe ansatz equations for the spectral parameters of the chosen algebra elements, which ensure that they indeed work as creation or annihilation operators.
In contrast to the first one, the solution via the Bethe ansatz method does not necessarily give the complete set of eigenvectors. There are a lot of discussions about the completeness of the Bethe ansatz method, but so far neither a proof, nor a correct counterexample has been found [19].
In both cases, the classical and the quantum mechanical, integrability is based on certain algebraic structures. Therefore the question arises, whether these can be constructed systematically. In the classical and the first quantum mechanical case this means that commutative algebras with a definite number of generators have to be constructed. Physically significant representations will then lead to integrable models. Of course there are numerous ways of constructing commutative algebras. In the following a formalism due to Ballesteros and Ragnisco [7] will be presented, which has shown to be extremely successful concerning the construction of classical and quantum mechanical integrable spin systems. It is exclusively based on the existence of a coproduct and a non-empty center and has an immediate generalization due to the author, enabling one to construct the class of so-called $\mathcal{B}$-partitioned spin systems [8], presented in the last section.
RTT algebras are usually constructed by $n^2 \times n^2$-matrix solutions $W(\lambda)$ of the so-called Yang-Baxter equation [20]:

$$\sum_{l_1,l_2,l_3} W(\mu-\lambda)^{il_1}_{jl_2} W(\mu)^{al_3}_{l_2 l} W(\lambda)^{l_3 b}_{l_1 k} = \sum_{l_1,l_2,l_3} W(\lambda)^{al_3}_{il_1} W(\mu)^{l_3 b}_{jl_2} W(\mu-\lambda)^{l_1 k}_{l_2 l} \tag{42}$$

Given such a solution, it can be interpreted as an $n \times n$-matrix of $n \times n$-matrices. As such, the index pairs (a, l_3) and (l_3, b) in (42) can be understood as the indices of operators in a certain basis. The index pairs (i, l_1) and (j, l_2) or (l_2, l) and (l_1, k) respectively then realize a tensor product at which the matrix $W(\mu-\lambda)$ acts in the sense of normal matrix multiplication. Hence an RTT algebra can be read off from a solution of the Yang-Baxter equation, just written in terms of the product structure on the two-fold tensorial product, which was defined in the preceding chapter. Although in all further theorems, propositions etc. the more compact notation in terms of the normal matrix multiplication will be used, some of the calculations will be done considering the product structure on the tensor product, because this avoids big matrices.
Drinfel´d constructed a theory concerned with the solution of the Yang-Baxter equation, which led to the development of Quantum Groups. This theory requires a so-called quasitriangular bialgebra, which is a quasicocommutative bialgebra together with two additional postulates on the coproduct:

$$(\Delta \otimes \mathrm{id})(R) = R^{13} R^{23} \tag{43}$$

$$(\mathrm{id} \otimes \Delta)(R) = R^{13} R^{12} \tag{44}$$

The terms on the right hand side are defined exactly the same way as in the classical case (26). A short calculation shows that the universal R matrix satisfies what is usually referred to as the Quantum Yang-Baxter equation

$$\begin{aligned} R^{12}R^{13}R^{23} &= R^{12}\,(\Delta\otimes\mathrm{id})\,(R) \\ &= (\Delta^{op}\otimes\mathrm{id})\,(R)R^{12} \\ &= (\tau\otimes\mathrm{id})\,(\Delta\otimes\mathrm{id})\,(R)R^{12} \\ &= (\tau\otimes\mathrm{id})\left(R^{13}R^{23}\right)R^{12} = R^{23}R^{13}R^{12}, \end{aligned}$$

where (43) has been applied. Using the second relation (44), the calculation starts from the right hand side of the equation. The fact that quasitriangularity postulates both, (43) and (44), results from further considerations, which are of no interest to this thesis.
Applying an arbitrary representation to the Quantum Yang-Baxter equation, a solution of the Yang-Baxter equation (42) results. In this thesis an approach of Drinfel´d [9], avoiding the Yang-Baxter equation and constructing RTT algebras directly from quasicocommutative bialgebras, will be used. The benefit is complete analogy with the formalism of Ballesteros and Ragnisco and a maximum of generality, because the relations (43) and (44) are dropped (Figure 2).

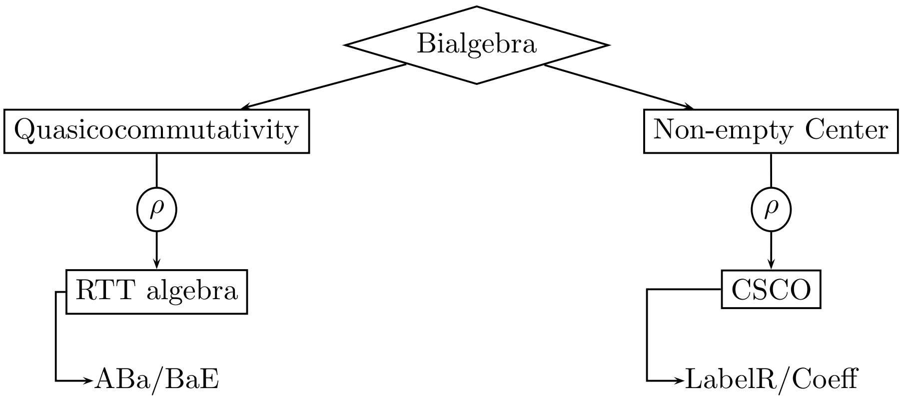

Figure 2: Schematic sketch of the theory´s structure (Abbreviations: **A**lgebraic **B**ethe **a**nsatz, **B**ethe **a**nsatz **E**quations, **C**omplete **S**ets of **C**ommuting **O**perators, **L**abel **R**ange, **Coeff**icients)

Obviously both of the branches, consisting of an algebraic background and a method to diagonalize a Hamiltonian, have certain advantages and disadvantages in exactly opposite parts. The formalism of Ballesteros and Ragnisco does not give complete sets of mutually commuting operators necessarily but if one is able to solve the problem, the solution is definitely complete. Drinfel´d´s construction of RTT algebras however definitely works, but an algebraic Bethe ansatz is not necessarily complete.
This stresses the clearly pragmatic level on which all the associated considerations

work. It is important to realize this, because there are approaches deeply differing from that, trying to work on in full rigor. Often resulting problems are simply ignored.

3.1. Central elements, coproducts and integrability

Let $(\mathcal{A}, m, \Delta)$ be a bialgebra and $(\mathcal{A}, m, \Delta, \{.,.\})$ a Poisson bialgebra. The basic idea of the following considerations is to use central elements, either with respect to the product in the first or with respect to the Poisson bracket in the second case, to construct commutative subsets of the N-fold tensorial product of the respective algebra. Suitable representations of these algebras then correspond either to quantum or to classical integrable models with N degrees of freedom, provided the additional conditions of the above definitions are satisfied.
The necessary maps $\mathcal{A} \to \otimes_{i=1}^{N} \mathcal{A}$ can either be constructed using the coproduct or be defined on their own. Both of these approaches will be discussed in the following.

3.1.1. Higher coproducts

Following Ballesteros and Ragnisco [7], let $\Delta^{(k)} : \mathcal{A} \to \otimes_{i=1}^{k} \mathcal{A}$ be the usual k-th coproduct, recurrently defined for the case $k > 1$ by

$$\Delta^{(k)} = \left(\Delta^{(2)} \otimes \mathrm{id}^{(k-2)}\right) \circ \Delta^{(k-1)}, \tag{45}$$

where $\mathrm{id}^{(k-2)}$ denotes the $(k-2)$-fold tensorial product of the identity map and $\Delta^{(2)}$ is the coproduct of the algebra. Let $\Delta^{(1)}$ furthermore defined to be the identity map.
It is crucial for all further considerations that any higher coproduct can be decomposed in arbitrary lower ones and that it is an algebra homomorphism or a Poisson map respectively, just like the usual one:

LEMMA 3-1. *Let $(\mathcal{A}, m, \Delta)$ be a bialgebra. Then the following identity holds for all $k > 1$ and $m = 1, \ldots, (k-1)$* [6]*:*

$$\Delta^{(k)} = \left(\Delta^{(m)} \otimes \Delta^{(k-m)}\right) \circ \Delta^{(2)} \tag{46}$$

Proof. The identity is shown by induction. Obviously it holds for $k = 2$, where no further calculation is necessary, as m can only be equal to one. This is not the case within the conclusion $k \to k+1$, where $m = 1, \ldots, k$ and the equality

[6] Do not mix up this index with the product!

has to be shown for all the associated decompositions. A direct calculation gives the first part:

$$\begin{aligned}
\Delta^{(k+1)} &= \left(\Delta^{(2)} \otimes \mathrm{id}^{(k-1)}\right) \circ \Delta^{(k)} \\
&= \left(\Delta^{(2)} \otimes \mathrm{id}^{(k-1)}\right) \circ \left(\Delta^{(n)} \otimes \Delta^{(k-n)}\right) \circ \Delta^{(2)} \\
&= \left(\left(\left(\Delta^{(2)} \otimes \mathrm{id}^{(n-1)}\right) \circ \Delta^{(n)}\right) \otimes \Delta^{(k-n)}\right) \circ \Delta^{(2)} \\
&= \left(\Delta^{(n+1)} \otimes \Delta^{(k-n)}\right) \circ \Delta^{(2)} \\
&= \left(\Delta^{(n+1)} \otimes \Delta^{((k+1)-(n+1))}\right) \circ \Delta^{(2)}
\end{aligned}$$

As $n = 1, \ldots, (k-1)$, the identity has been shown for $m = n+1 = 2, \ldots, k$. Hence it remains to proof the case $m = 1$

$$\Delta^{(k+1)} = \left(id \otimes \Delta^{(k)}\right) \circ \Delta^{(2)},$$

for which, besides the assumption, the coassociativity is used:

$$\begin{aligned}
\Delta^{(k+1)} &= \left(\Delta^{(2)} \otimes \mathrm{id}^{(k-1)}\right) \circ \Delta^{(k)} \\
&= \left(\Delta^{(2)} \otimes \mathrm{id} \otimes \mathrm{id}^{(k-2)}\right) \circ \Delta^{(k)} \\
&= \left(\Delta^{(2)} \otimes \mathrm{id} \otimes \mathrm{id}^{(k-2)}\right) \circ \left(\Delta^{(2)} \otimes \Delta^{(k-2)}\right) \circ \Delta^{(2)} \\
&= \left(\left(\left(\Delta^{(2)} \otimes \mathrm{id}\right) \circ \Delta^{(2)}\right) \otimes \Delta^{(k-2)}\right) \circ \Delta^{(2)} \\
&= \left(\left(\left(\mathrm{id} \otimes \Delta^{(2)}\right) \circ \Delta^{(2)}\right) \otimes \Delta^{(k-2)}\right) \circ \Delta^{(2)} \\
&= \left(\mathrm{id} \otimes \Delta^{(2)} \otimes \mathrm{id}^{(k-2)}\right) \circ \left(\Delta^{(2)} \otimes \Delta^{(k-2)}\right) \circ \Delta^{(2)} \\
&= \left(\mathrm{id} \otimes \Delta^{(2)} \otimes \mathrm{id}^{(k-2)}\right) \circ \Delta^{(k)} \\
&= \left(\mathrm{id} \otimes \Delta^{(2)} \otimes \mathrm{id}^{(k-2)}\right) \circ \left(\mathrm{id} \otimes \Delta^{(k-1)}\right) \circ \Delta^{(2)} \\
&= \left(\mathrm{id} \otimes \left(\left(\Delta^{(2)} \otimes \mathrm{id}^{(k-2)}\right) \circ \Delta^{(k-1)}\right)\right) \circ \Delta^{(2)} \\
&= \left(\mathrm{id} \otimes \Delta^{(k)}\right) \circ \Delta^{(2)}
\end{aligned}$$

□

LEMMA 3-2. *Let $(\mathcal{A}, m, \Delta, \{.,.\})$ be a (Poisson) bialgebra. Any higher coproduct is an algebra homomorphism or a Poisson map respectively*

$$\left[\Delta^{(k)}(X), \Delta^{(k)}(Y)\right\} = \Delta^{(k)}\left([X, Y\}\right), \tag{47}$$

where $X, Y \in \mathcal{A}$.

Proof. The identity is again shown by induction. The cases $k = 1, 2$ hold by definition in the algebra homomorphism as well as the Poisson case. The conclusion $k-1 \to k$ first should be made for the algebra homomorphism property, using Lemma 3-1 and Sweedler´s notation.

$$\begin{aligned}
\Delta^{(k)}(X)\Delta^{(k)}(Y) &= \left(\left(\Delta^{(k-1)} \otimes \mathrm{id}\right)\left(\Delta^{(2)}(X)\right)\right)\left(\left(\Delta^{(k-1)} \otimes \mathrm{id}\right)\left(\Delta^{(2)}(Y)\right)\right) \\
&= \sum_{ij} \left(\Delta^{(k-1)}\left(X_{1i}\right) \otimes X_{2i}\right)\left(\Delta^{(k-1)}\left(Y_{1j}\right) \otimes Y_{2j}\right) \\
&= \sum_{ij} \Delta^{(k-1)}\left(X_{1i}Y_{1j}\right) \otimes X_{2i}Y_{2j} \\
&= \left(\Delta^{(k-1)} \otimes \mathrm{id}\right)\left(\sum_{ij} X_{1i}Y_{1j} \otimes X_{2i}Y_{2j}\right) \\
&= \left(\Delta^{(k-1)} \otimes \mathrm{id}\right) \circ \Delta^{(2)}(XY) \\
&= \Delta^{(k)}(XY)
\end{aligned}$$

The Poisson map case follows the same way:

$$\begin{aligned}
&\left\{\Delta^{(k)}(X), \Delta^{(k)}(Y)\right\} = \left\{\left(\Delta^{(k-1)} \otimes \mathrm{id}\right)\left(\Delta^{(2)}(X)\right), \left(\Delta^{(k-1)} \otimes \mathrm{id}\right)\left(\Delta^{(2)}(Y)\right)\right\} \\
&= \left\{\left(\Delta^{(k-1)} \otimes \mathrm{id}\right)\left(\sum_i X_{1i} \otimes X_{2i}\right), \left(\Delta^{(k-1)} \otimes \mathrm{id}\right)\left(\sum_j Y_{1j} \otimes Y_{2j}\right)\right\} \\
&= \sum_{ij} \left\{\Delta^{(k-1)}(X_{1i}) \otimes X_{2i}, \Delta^{(k-1)}(Y_{1j}) \otimes Y_{2j}\right\} \\
&= \sum_{ij} \left\{\Delta^{(k-1)}(X_{1i}), \Delta^{(k-1)}(Y_{1j})\right\} \otimes X_{2i}Y_{2j} \\
&\quad + \left(\Delta^{(k-1)}(X_{1i})\Delta^{(k-1)}(Y_{1j})\right) \otimes \{X_{2i}, Y_{2j}\} \\
&= \sum_{ij} \Delta^{(k-1)}\left(\{X_{1i}, Y_{1j}\}\right) \otimes (X_{2i}Y_{2j}) + \Delta^{(k-1)}(X_{1i}Y_{1j}) \otimes \{X_{2i}, Y_{2j}\} \\
&= \sum_{ij} \left(\Delta^{(k-1)} \otimes \mathrm{id}\right)\left(\{X_{1i}, Y_{1j}\} \otimes X_{2i}Y_{2j} + X_{1i}Y_{1j} \otimes \{X_{2i}, Y_{2j}\}\right) \\
&= \sum_{ij} \left(\Delta^{(k-1)} \otimes \mathrm{id}\right)\left(\{X_{1i} \otimes X_{2i}, Y_{1j} \otimes Y_{2j}\}\right) \\
&= \left(\Delta^{(k-1)} \otimes \mathrm{id}\right)\left(\left\{\sum_i X_{1i} \otimes X_{2i}, \sum_j Y_{1j} \otimes Y_{2j}\right\}\right) \\
&= \left(\Delta^{(k-1)} \otimes \mathrm{id}\right)\left(\left\{\Delta^{(2)}(X), \Delta^{(2)}(Y)\right\}\right) \\
&= \left(\left(\Delta^{(k-1)} \otimes \mathrm{id}\right) \circ \Delta^{(2)}\right)\left(\{X, Y\}\right) = \Delta^{(k)}\left(\{X, Y\}\right)
\end{aligned}$$

□

If the higher coproducts are embedded in $\otimes_{i=1}^{N} \mathcal{A}$ by

$$\Delta^{(k)} \otimes 1^{(N-k)}, \tag{48}$$

it is possible with Lemma 3-1 and Lemma 3-2 to prove the very important first theorem, which is the key point of [7]:

THEOREM 3-1. *Let $(\mathcal{A}, m, \Delta, \{.,.\})$ be a (Poisson) bialgebra with $C \in Z(\mathcal{A})$. Any two higher coproducts of C commute with each other and with the highest coproduct of any $X \in \mathcal{A}$:*

$$\big[\Delta^{(i)}(C), \Delta^{(j)}(C)\big\} = \big[\Delta^{(k)}(C), \Delta^{(N)}(X)\big\} = 0 \tag{49}$$

Proof. First it has to be shown that any higher coproduct commutes with the highest of any element X. This can be done for the commutator and the Poisson case simultaneously:

$$\begin{aligned}
\big[\Delta^{(k)}(C), \Delta^{(N)}(X)\big\} &= \big[\Delta^{(k)}(C) \otimes 1^{(N-k)}, \Delta^{(N)}(X)\big\} \\
&= \big[\Delta^{(k)}(C) \otimes 1^{(N-k)}, \big(\Delta^{(k)} \otimes \Delta^{(N-k)}\big) \circ \Delta^{(2)}(X)\big\} \\
&= \sum_j \big[\Delta^{(k)}(C) \otimes 1^{(N-k)}, \Delta^{(k)}\left(X_{1j}\right) \otimes \Delta^{(N-k)}\left(X_{2j}\right)\big\} \\
&= \sum_j \big[\Delta^{(k)}(C), \Delta^{(k)}(X_{1j})\big\} \otimes \Delta^{(N-k)}(X_{2j}) \\
&= \sum_j \Delta^{(k)}\left([C, X_{1j}\}\right) \otimes \Delta^{(N-k)}\left(X_{2j}\right) \\
&= 0
\end{aligned}$$

Writing down the embedding (48) explicitely, the left hand side of the first identity reads:

$$\big[\Delta^{(i)}(C), \Delta^{(j)}(C)\big\} \otimes 1^{(N-j)}$$

Assuming $j > i$, which of course causes no loss of generality, the application of the second identity to the subspace $\otimes_{i=1}^{j} \mathcal{A}$ gives the first one.

□

Theorem 3-1 gives rise to $(N+1)$ commuting elements in $\otimes_{i=1}^{N} \mathcal{A}$. The physical significance now depends on the realization of $\otimes_{i=1}^{N} \mathcal{A}$. Let $\mathcal{P}$ denote the phase space of a classical system and $\mathcal{H}$ the Hilbert space of a quantum mechanical one. The most important case for this thesis results if $\mathcal{A}$ is realized as the set of functions $F(\mathcal{P})$ on a phase space or as the set of operators $O(\mathcal{H})$ on a Hilbert space. This would correspond to an N particle system, each with one degree

of freedom, hence exactly the spin situtation. Therefore only realizations which map the respective central element to a constant could give rise to an integrable system.
For that, in the classical case, the functions, resulting from the algebra elements, have to be functionally independent. This demand is satisfied for all $\Delta^{(i)}(C)$, as these are elements of different subspaces. Hence it remains to choose the element X correctly. In quantum mechanical realizations the situation is not that clear, because, as mentioned above, the set of N commuting operators has not to be complete necessarily. However, in many important situations it works out very well.

3.1.2. "Coproducts" on partition trees

Although Theorem 3-1 is very general and has a lot of important physical applications, it has a significant restriction. This becomes clear if the standard coproduct (10) is considered. If for example $k = 3$, the definition (45) gives:

$$\Delta^{(3)}(X_i) = X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1 + 1 \otimes 1 \otimes X_i$$

This structure is what should be called "monotonous" from now on. Obviously (45) does not permit to construct "non-monotonous" maps $\mathcal{A} \to \otimes_{i=1}^{N} \mathcal{A}$ like for example:

$$X_i \to X_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes X_i$$

It will become clear later on that it is extremely profitable to generalize Theorem 3-1 to such non-monotonous structures. This essentially means that the maps $\mathcal{A} \to \otimes_{i=1}^{N} \mathcal{A}$ cannot be induced by a coproduct in the sense of (45) anymore. A new definition is necessary:

DEFINITION 3-1. *Let M be a set of integers. Let furthermore N be an integer greater or equal to the greatest element of M. If $(\mathcal{A}, m)$ is an algebra with generators $\{X_i\}$, Δ^M, which should be called pseudo-coproduct, is defined on the generators and extended as an algebra homomorphism*

$$\Delta^M(X_i) = \sum_{j=1}^{|M|} \Delta_j^M(X_i), \tag{50}$$

where Δ_j^M is the element of $\otimes_{i=1}^{N} \mathcal{A}$ with X_i at the position given by the j-th element of M and 1 otherwise.

Consider for example $M = \{1, 3\}$ and $N = 3$, then Δ^M is given by:

$$\Delta^M(X_i) = X_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes X_i$$

It becomes clear with the above example that if $N := \{1, \ldots, N\}$, the map induced recurrently by the standard coproduct and the one given by (50) are the same $\Delta^N = \Delta^{(N)}$. In the following the ambigous use of N as an integer and the above set of intergers will be kept. It will become completely clear from the context which of the both is meant.
Now a version of Lemma 3-2 corresponding to (50) has to be proven. For that the following easy lemma is necessary:

LEMMA 3-3. *Let* $(\mathcal{A}, m, \{.,.\})$ *a (Poisson) algebra and* I, J *two sets of integers with* $I \cap J = \emptyset$. *Then the pseudo-coproduct associated with* I *and* J *of any two algebra elements* X, Y *commute with each other:*

$$\left[\Delta^I(X), \Delta^J(Y)\right\} = 0 \tag{51}$$

Proof. The pseudo-coproduct of an arbitrary element of the algebra is an element of the same subspace as the pseudo-coproduct of a generator. This means that any of the resulting terms of $\Delta^I(X)$ has nontrivial components at the same positions as the pseudo-coproduct of a generator. With $I \cap J = \emptyset$ the commutator case therefore results immediately.
It is clear from the definition that the Poisson bracket of an algebra element with the unit element is equal to zero:

$$L_Z(1) = L_Z(11) = L_Z(1) + L_Z(1) = 2L_Z(1) \Leftrightarrow L_Z(1) = 0$$

Therefore it results for every term of (51)

$$\{a \otimes 1, b \otimes c\} = \{a, b\} \otimes c + ab \otimes \{1, c\} = \{a, b\} \otimes c,$$

where $a, b \in \otimes_{i=1}^{(N-1)} \mathcal{A}$ and $c \in \mathcal{A}$. Repeating this decomposition $(N-1)$ times, the Poisson case of (51) results.

□

It should be notized that in Lemma 3-3 no coalgebra structure is postulated. This becomes necessary first within the proof of the following lemma:

LEMMA 3-4. *Let* $(\mathcal{A}, m, \Delta, \{.,.\})$ *be a Poisson bialgebra with generators* $\{X_i\}$. *Any pseudo-coproduct is a Poisson map:*

$$\left\{\Delta^M(X), \Delta^M(Y)\right\} = \Delta^M\left(\{X, Y\}\right) \tag{52}$$

Proof. The identity has to be proven on the generators first:

$$\begin{aligned}\left\{\Delta^N(X_i),\Delta^N(X_j)\right\} &= \left\{\Delta^I(X_i)+\Delta^J(X_i),\Delta^I(X_j)+\Delta^J(X_j)\right\}\\ &= \left\{\Delta^I(X_i),\Delta^I(X_j)\right\}+\left\{\Delta^J(X_i),\Delta^J(X_j)\right\}\end{aligned}$$

As $\Delta^N = \Delta^{(N)}$ and $\mathcal{A}$ is a Poisson bialgebra, it follows from Lemma 3-2 that:

$$\left\{\Delta^I(X_i),\Delta^I(X_j)\right\}+\left\{\Delta^J(X_i),\Delta^J(X_j)\right\}=\Delta^N\left(\{X_i,X_j\}\right)$$

The right hand side of this equation can be rewritten in terms of the pseudo-coproducts on the left hand side, using the structure constants of the Poisson bracket:

$$\begin{aligned}\Delta^N\left(\{X_i,X_j\}\right) &= \sum_k c_{ijk}\Delta^N\left(X_k\right)=\sum_k c_{ijk}\left[\Delta^I\left(X_k\right)+\Delta^J\left(X_k\right)\right]\\ &= \Delta^I\left(\{X_i,X_j\}\right)+\Delta^J\left(\{X_i,X_j\}\right)\end{aligned}$$

Combining the both identities gives:

$$\left(\left\{\Delta^I(X_i),\Delta^I(X_j)\right\}-\Delta^I\left(\{X_i,X_j\}\right)\right)+\left(\left\{\Delta^J(X_i),\Delta^J(X_j)\right\}-\Delta^J\left(\{X_i,X_j\}\right)\right)=0$$

As the Poisson bracket maps elements on certain subspaces on the same ones, it follows immediately that both of the above brackets have to be equal to zero. The highest coproduct can be decomposed in any way and hence the identity holds for every I, J.

It remains to show that the identity holds for any algebra element. For that it is convenient to start with one arbitrary element $X \in \mathcal{A}$. Using the same decomposition of algebra elements, as in the proof of Theorem 2-1 and the algebra homomorphism property of the pseudo-coproduct, it follows:

$$\left\{\Delta^I(X),\Delta^I(X_j)\right\}=\sum_k a^{i_1\dots i_k}\left\{\Delta^I\left(X_{i_1}\right)\dots\Delta^I\left(X_{i_k}\right),\Delta^I(X_j)\right\}$$

It results immediately from the definition of the Poisson bracket that an arbitrary product gives

$$L_Z\left(\prod_{i=1}^n a_i\right)=L_Z(a_1)\prod_{j=2}^n a_j+\sum_{k=2}^{n-1}\prod_{l=1}^{k-1}a_lL_Z(a_k)\prod_{m=k+1}^n a_m+\prod_{j=1}^{n-1}a_jL_Z(a_n) \qquad (53)$$

and with $a_l = \Delta^M\left(X_{i_l}\right)$ and $Z = \Delta^M\left(X_j\right)$, it results:

$$\begin{aligned}\left\{\Delta^M(X),\Delta^M(X_j)\right\} &= \sum_k a^{i_1\dots i_k}\left\{\Delta^M\left(X_{i_1}\right),\Delta^M\left(X_j\right)\right\}\prod_{\lambda=2}^k\Delta^M\left(X_{i_\lambda}\right)\\ &+ \sum_{l=2}^{k-1}\prod_{\lambda=1}^{l-1}\Delta^M\left(X_{i_\lambda}\right)\left\{\Delta^M\left(X_{i_l}\right),\Delta^M\left(X_j\right)\right\}\prod_{\mu=l+1}^k\Delta^M\left(X_{i_\mu}\right)\\ &+ \prod_{\lambda=1}^{k-1}\Delta^M\left(X_{i_\lambda}\right)\left\{\Delta^M\left(X_{i_k}\right),\Delta^M\left(X_j\right)\right\}\end{aligned}$$

Using the first part of the proof gives:

$$\begin{aligned}\left\{\Delta^M(X),\Delta^M(X_j)\right\} &= \sum_k a^{i_1\dots i_k}\Delta^M\left(\{X_{i_1},X_j\}\right)\prod_{\lambda=2}^{k}\Delta^M\left(X_{i_\lambda}\right)\\ &+ \sum_{l=2}^{k-1}\prod_{\lambda=1}^{l-1}\Delta^M\left(X_{i_\lambda}\right)\Delta^M\left(\{X_{i_l},X_j\}\right)\prod_{\mu=l+1}^{k}\Delta^M\left(X_{i_\mu}\right)\\ &+ \prod_{\lambda=1}^{k-1}\Delta^M\left(X_{i_\lambda}\right)\Delta^M\left(\{X_{i_k},X_j\}\right)\end{aligned}$$

Now again the algebra homomorphism property can be used, finishing the calculation:

$$\left\{\Delta^M(X),\Delta^M(X_j)\right\} = \sum_k a^{i_1\dots i_k}\Delta^M\left(\{X_{i_1},X_j\}\prod_{\lambda=2}^{k}X_{i_\lambda}\right)$$
$$+\sum_k a^{i_1\dots i_k}\Delta^M\left(\sum_{l=2}^{k-1}\prod_{\lambda=1}^{l-1}X_{i_\lambda}\left\{X_{i_l},X_j\right\}\prod_{\mu=l+1}^{k}X_{i_\mu}+\prod_{\lambda=1}^{k-1}X_{i_\lambda}\left\{X_{i_k},X_j\right\}\right)$$
$$=\sum_k a^{i_1\dots i_k}\Delta^M\left(\{X_{i_1}\dots X_{i_k},X_j\}\right)=\Delta^M\left(\{X,X_j\}\right)$$

The calculation for the second element is exactly the same.

□

Now the problem of generalizing Theorem 3-1 to pseudo-coproducts can be tackled by defining the notion of a partition tree:

DEFINITION 3-2. *A partition tree $\mathcal{B}$ over a finite set $\{1,\dots,N\}$ is a set of subsets of $\{1,\dots,N\}$ satisfying*

a. $\emptyset\notin\mathcal{B}$ *and* $\{1,\dots,N\}\in\mathcal{B}$,

b. *for all* $M,N\in\mathcal{B}$ *either* $M\cap N=\emptyset$ *or* $M\subset N$ *or* $N\subset M$,

c. *for all* $M\in\mathcal{B}$ *with* $|M|>1$ *there exist* $M_1,M_2\in\mathcal{B}$ *such that* $M=M_1\dot\cup M_2$.

Due to (b), the branches M_1 and M_2 starting from M are unique. The set $\{1,\dots,N\}$ is called the root, whereas singletons serve as leaves. Elements of the same size form a level of the partition tree. It follows from (b) that the intersection of any two elements I,J on the same level is empty $I\cap J=\emptyset$. Due to (c), the same holds for any two elements on different branches. An example of a partition tree is given in (Figure 3).

Now it will be shown that Theorem 3-1 generalizes to (50) on partition trees,

$$\begin{tikzcd}
\{1\} & & \{3\} & & \{2\} & & \{4\} \\
& \{1,3\} \arrow[ul] \arrow[ur] & & & & \{2,4\} \arrow[ul] \arrow[ur] & \\
& & & \{1,2,3,4\} \arrow[ull] \arrow[urr] & & &
\end{tikzcd}$$

Figure 3: An example of a partition tree with N=4

which can be seen as a central result of this thesis:

THEOREM 3-2. *Let $(\mathcal{A}, \Delta, \{.,.\})$ a (Poisson bi-)algebra with generators $\{X_i\}$ and a non-empty center $Z(\mathcal{A})$. Let furhtermore $\mathcal{B}$ be a partition tree. If $C \in Z(\mathcal{A})$ and $N := \{1, \ldots, N\}$, it follows for all $I, J, K \in \mathcal{B}$ and $X \in \mathcal{A}$:*

$$\left[\Delta^I(C), \Delta^J(C)\right\} = \left[\Delta^K(C), \Delta^N(X)\right\} = 0 \tag{54}$$

Proof. The second identity is again proven on the generators first, using Lemma 3-3 and Lemma 3-4:

$$\begin{aligned}
\left[\Delta^K(C), \Delta^N(X_i)\right\} &= \left[\Delta^K(C), \Delta^K(X_i) + \Delta^{N\setminus K}(X_i)\right\} \\
&= \left[\Delta^K(C), \Delta^K(X_i)\right\} + \left[\Delta^K(C), \Delta^{N\setminus K}(X_i)\right\} \\
&= \Delta^K\left([C, X_i\}\right) \\
&= 0
\end{aligned}$$

The rest results exactly the same way as in the proof of Lemma 3-4, where in (53) $a_l = \Delta^N(X_{i_l})$ and $Z = \Delta^K(C)$. It should be stressed that the identity follows exclusively from the properties of the pseudo-coproducts and $K \subset N$. Indeed it holds for all sets I, J with $I \subset J$.
Now the first identity can be proven and the partition tree enters the game: If I and J are on different branches or on the same level, it holds due to Lemma 3-3, because their intersection is empty. For all the other cases it can be assumed that $I \subset J$ without any restriction and therefore it results from the second identity.

□

It can be shown by induction that $|\mathcal{B}| = 2N - 1$ [8] so that Theorem 3-2 seems to give $2N$ commuting elements. But the N elements resulting from the lowest level of the tree trivially commute with each other so that only one arbitrary element remains. Therefore the approach, just like the original one of Ballesteros and Ragnisco, gives $(N + 1)$ commuting elements and the physical significance again depends on the realization. A very interesting example for such a realization gives the so-called $\mathcal{B}$-partitioned spin systems, which will be presented in the next section.

3.1.3. Commutative subsets from coproducts and pseudo-coproducts in comparison

A partition tree which contains all singletons and all ordered subsets $\{1, \ldots, n\}$ for all $n \leq N$ is called monotonous. Considering for example $N = 3$, the following tree is monotonous:

$$\mathcal{B} = \{\{1,2,3\}, \{1,2\}, \{1\}, \{2\}, \{3\}\}$$

Let the coproduct of Theorem 3-1 now be defined on the generators by the standard coproduct (10) and extended as an algebra homomorphism to the whole algebra. If in this case the same algebra and a monotonous partition tree is considered, Theorem 3-1 and Theorem 3-2 give rise to exactly the same set of commuting elements. This can be illustrated using the above partition tree and remembering the embedding used in Theorem 3-1:

$$\begin{aligned}
\Delta^{\{1,2,3\}}(X_i) &= X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1 + 1 \otimes 1 \otimes X_i = \Delta^{(3)}(X_i) \\
\Delta^{\{1,2\}}(X_i) &= X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1 = \Delta^{(2)}(X_i) \otimes 1 \\
\Delta^{\{1\}}(X_i) &= X_i \otimes 1 \otimes 1 = \Delta^{(1)}(X_i) \otimes 1 \otimes 1
\end{aligned}$$

This justifies the notion of monotonous structures, used above.

3.2. RTT algebras from quasicocommutative bialgebras

In the last section it has been shown that classical as well as quantum integrable models, in the sense of complete sets of mutually commuting operators, can be deduced from bialgebra structures. In this section it will be explained how algebras relevant for the algebraic Bethe ansatz could be derived based on the same structure.

The crucial property of the bialgebras in Theorem 3-1 and Theorem 3-2 was a non-empty center. In the context of RTT algebras it is the quasicocommutativity, defined in the first chapter. Before proving the associated theorem, a small lemma, important for the following considerations, has to be shown:

LEMMA 3-5. *Let $(\mathcal{A}, m, \Delta)$ be a bialgebra and $\rho : \mathcal{A} \to \mathrm{Mat}(\mathbb{C}, n)$ a representation on the set of $n \times n$-matrices with complex entries. The representation ρ is described by an $n \times n$-matrix T of dual elements $t_{ij} \in \mathcal{A}^*$, if and only if the coproduct $\bar{\Delta}$ of the dual bialgebra is given by*

$$\bar{\Delta}(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}. \tag{55}$$

Proof. "$\Rightarrow$": As ρ is described by T, it satisfies $T(XY) = T(X)T(Y)$ for all

$X, Y \in A$, where on the left hand side the product on $\mathcal{A}$ and on the right hand side the one on $\mathrm{Mat}(\mathbb{C}, n)$ is considered. This is equivalent to

$$t_{ij}(XY) = \sum_{k=1}^{n} t_{ik}(X) t_{kj}(Y).$$

It follows

$$\begin{aligned}
\langle \bar{\Delta}\left(t_{ij}\right) | X \otimes Y \rangle &= \langle t_{ij} | XY \rangle = t_{ij}(XY) \\
&= \sum_{k=1}^{n} t_{ik}(X) t_{kj}(Y) \\
&= \sum_{k=1}^{n} \langle t_{ik} \otimes t_{kj} | X \otimes Y \rangle,
\end{aligned}$$

which gives (55).
"$\Leftarrow$": The proof follows with nearly the same calculation as above

$$\begin{aligned}
t_{ij}(XY) &= \langle t_{ij} | XY \rangle = \langle \bar{\Delta}\left(t_{ij}\right) | X \otimes Y \rangle \\
&= \sum_{k=1}^{n} \langle t_{ik} \otimes t_{kj} | X \otimes Y \rangle \\
&= \sum_{k=1}^{n} t_{ik}(X) t_{kj}(Y),
\end{aligned}$$

which is equivalent to $T(XY) = T(X)T(Y)$, completning the proof.

□

Now the crucial theorem [9, 21] can be proven:

THEOREM 3-3. *Let $(\mathcal{A}, m, \Delta)$ be a quasicocommutative bialgebra and $\rho : \mathcal{A} \to \mathrm{Mat}(\mathbb{C}, n)$ a representation, described by an $n \times n$-matrix of dual elements $t_{ij} \in A^*$. These elements satisfy the RTT relations*

$$R^{\rho} T_1 T_2 = T_2 T_1 R^{\rho}, \tag{56}$$

where again $R^{\rho} = (\rho \otimes \rho)(R)$ and T_1 and T_2 are defined by (41).

Proof. Using Sweedler´s notation both sides of (56) can be written down in components

$$\left[R^{\rho} T_1 T_2\right]^{ij}_{kl} = \sum_{m,n,q} t_{im}\left(R_{1q}\right) t_{mj} t_{kn}\left(R_{2q}\right) t_{nl} \tag{57}$$

$$\left[T_2 T_1 R^{\rho}\right]^{ij}_{kl} = \sum_{m,n,q} t_{km} t_{ml}\left(R_{2q}\right) t_{in} t_{nj}\left(R_{1q}\right), \tag{58}$$

where the compact notation in terms of the usual matrix product has been transferred into the tensor product form again. Now the right hand side of (18) has to be used together with Sweedler´s notation and Lemma 3-5:

$$\begin{aligned}
(t_{ij}\otimes t_{kl})\,(R\Delta(X)) &= \sum_{p,q} t_{ij}\,(R_{1p}X_{1q})\,t_{kl}\,(R_{2p}X_{2q}) \\
&= \sum_{p,q,m,n} t_{im}\,(R_{1p})\,t_{mj}\,(X_{1q})\,t_{kn}\,(R_{2p})\,t_{nl}\,(X_{2q}) \\
&= \sum_{q,m,n} t_{im}\,(R_{1p})\,t_{kn}\,(R_{2p})\,\langle t_{mj}\otimes t_{nl}|\Delta(X)\rangle \\
&= \sum_{q,m,n} t_{im}\,(R_{1p})\,t_{kn}\,(R_{2p})\,\langle t_{mj}t_{nl}|X\rangle \\
&= \left(\sum_{q,m,n} t_{im}\,(R_{1p})\,t_{kn}\,(R_{2p})\,t_{mj}t_{nl}\right)(X)
\end{aligned} \tag{59}$$

The left hand side gives:

$$\begin{aligned}
(t_{ij}\otimes t_{kl})\,(\Delta^{op}(X)R) &= \sum_{p,q} t_{ij}\,(X_{2p}R_{1q})\,t_{kl}\,(X_{1p}R_{2q}) \\
&= \sum_{p,q,m,n} t_{im}\,(X_{2p})\,t_{mj}\,(R_{1q})\,t_{kn}\,(X_{1p})\,t_{nl}\,(R_{2q}) \\
&= \sum_{q,m,n} t_{mj}\,(R_{1q})\,t_{nl}\,(R_{2q})\,\langle t_{im}\otimes t_{kn}|\Delta(X)\rangle \\
&= \sum_{q,m,n} t_{mj}\,(R_{1q})\,t_{nl}\,(R_{2q})\,\langle t_{im}t_{kn}|X\rangle \\
&= \left(\sum_{q,m,n} t_{mj}\,(R_{1q})\,t_{nl}\,(R_{2q})\,t_{im}t_{kn}\right)(X)
\end{aligned} \tag{60}$$

Interchanging the indices in (59) and (60) gives (57) and (58), which finishes the proof.

□

The following corollary is crucial for the subsequent applications:

COROLLARY 3-1. *Let $(\mathcal{A}, m, \Delta)$ be a quasicocommutative bialgebra with two representations $\rho : \mathcal{A} \to \mathrm{Mat}(\mathbb{C}, n)$ and $\tilde{\rho} : \mathcal{A} \to \mathrm{Mat}(\mathbb{C}, n)$, described by two $n \times n$-matrices of dual elements T and $\tilde{T}$. Let furthermore $(\rho\otimes\rho)(R) = (\tilde{\rho}\otimes\tilde{\rho})(R)$ and $\left[\tilde{T}_{1/2}, T_{2/1}\right] = 0$. Then a new RTT algebra results:*

$$R^{\rho}T_1\tilde{T}_1T_2\tilde{T}_2 = T_2\tilde{T}_2T_1\tilde{T}_1R^{\rho} \tag{61}$$

Proof. Using Theorem 3-3 the proof results immediately:

$$\begin{aligned} R^\rho T_1 \tilde{T}_1 T_2 \tilde{T}_2 &= R^\rho T_1 T_2 \tilde{T}_1 \tilde{T}_2 \\ &= T_2 T_1 R^\rho \tilde{T}_1 \tilde{T}_2 \\ &= T_2 T_1 \tilde{T}_2 \tilde{T}_1 R^\rho = T_2 \tilde{T}_2 T_1 \tilde{T}_1 R^\rho \end{aligned}$$

□

To construct integrable quantum systems from Theorem 3-3, the dual elements t_{ij} have to be operators on the respective Hilbert space. Due to the selfduality of the bialgebra, their product is given by the coproduct of the algebra. This means that the action of the product of any two elements t_{ij} on the Hilbert space has to be defined as the normal operator product. This corresponds to a representation of the RTT algebra and if such a representation exists, it has to be checked individually.

It might seem a bit surprising that Theorem 3-3 works with parameter independent representations. But it will become clear that $T(\lambda)$, regarded as representation, is parameter independent.

4. Integrable Heisenberg systems from bialgebras

In this chapter concrete physical systems will be deduced from abstract bialgebra structures using Theorem 3-2 and Theorem 3-3. As mentioned before, the first one will be used to derive the class of so-called $\mathcal{B}$-partitioned spin systems, which were originally constructed by Schmidt and Steinigeweg [8]. The second example is the well-known XXX-S=1/2-Heisenberg chain. It will be solved by algebraic Bethe ansatz based on Theorem 3-3. Afterwards the original coordinate Bethe ansatz will be presented and both of the solutions will be compared with each other.

4.1. $\mathcal{B}$-partitioned spin systems

A $\mathcal{B}$-partitioned spin system is a Heisenberg model with an interaction defined on a partition tree $\mathcal{B}$ [8]:

DEFINITION 4-1. *Let $J : \mathcal{B} \rightarrow \mathbb{R}$ and $M_{ij} \in \mathcal{B}$ be the smallest set containing the singletons i and j. Then the Hamiltonian of a $\mathcal{B}$-partitioned spin system is defined as:*

$$H = \sum_{i<j} J\left(M_{ij}\right) \vec{\alpha}_i \vec{\alpha}_j \tag{62}$$

In the following H should be considered as a quantum mechanical or a classical system. Accordingly, the vector $\vec{\alpha}_i$ either denotes a spin operator $\vec{S}_i$ acting on the i-th Hilbertspace $\mathcal{H}_i$ or the coordinate function $\vec{s}_i$ of a classical spin vector "acting on" the i-th phase space $\mathcal{P}_i = S^2$.
To use Theorem 3-2, (Poisson) bialgebras $(\mathcal{A}, m, \Delta, \{.,.\})$ have to be defined, which can be realized as $O(\mathcal{H})$ and $F(\mathcal{P})$:

DEFINITION/PROPOSITION 4-2. *Let $(\mathcal{A}, m, \Delta, \{.,.\})$ be the (Poisson) bialgebra with three generators X_1, X_2, X_3, a (non-) commutative product and the "defining relations"*

$$[X_i, X_j\} = (i) \sum_{k=1}^{3} \epsilon_{ijk} X_k, \tag{63}$$

where ϵ_{ijk} denotes the Levi-Civita symbol and (63) should be extended as a derivation to the whole algebra. The coproduct is given by (10) and extended as an algebra homomorphism to the whole algebra.

Obviously the definition is a little informal, as two different algebras are defined at the same time due to their similarity. The first one, corresponding to the quantum case, is the universal enveloping algebra of the $\mathrm{sl}(2, \mathbb{C})$, just given by generators different from the ones in the definition of the Yangian. In this case the product is non-commutative and (63), which has to be taken with the i in this case, really gives a set of defining relations. This is not the case if the algebra is considered to be a commutative Poisson bialgebra. In that situation the "defining relations" just give an additional structure on the algebra and hence are no real relations. In the following all the considerations will be made for both of the cases simultanously, always considering the correct algebra. Now the proof can be given:

Proof. It is convenient to start with the pure bialgebra case. Hence it has to be shown that the standard coproduct indeed extends as an algebra homomorphism:

$$\begin{aligned}
[\Delta(X_i), \Delta(X_j)] &= \Delta(X_i)\Delta(X_j) - \Delta(X_j)\Delta(X_i) \\
&= X_iX_j \otimes 1 + 1 \otimes X_iX_j + X_i \otimes X_j + X_j \otimes X_i \\
&- X_jX_i \otimes 1 - 1 \otimes X_jX_i - X_i \otimes X_j - X_j \otimes X_i \\
&= [X_i, X_j] \otimes 1 + 1 \otimes [X_i, X_j] \\
&= \sum_k \epsilon_{ijk} \left(X_k \otimes 1 + 1 \otimes X_k\right) \\
&= \sum_k \epsilon_{ijk} \Delta(X_k) \\
&= \Delta\left([X_i, X_j]\right),
\end{aligned}$$

The extension to arbitrary algebra elements results immediately. Now the Poisson case can be considered, in case of which a commutative product has to be used

$$[\Delta(X_i), \Delta(X_j)] = 0 = \Delta(0) = \Delta\left([X_i, X_j]\right),$$

where the second equality results from the linearity of the coproduct:

$$\Delta(0) = \Delta(0+0) = 2\Delta(0) \Leftrightarrow \Delta(0) = 0$$

That the standard coproduct is a Poisson map and hence defines a Poisson bialgebra structure, will be shown on the generators first using (15):

$$\begin{aligned}
\left\{\Delta\left(X_i\right), \Delta\left(X_j\right)\right\} &= \{1 \otimes X_i + X_i \otimes 1, 1 \otimes X_j + X_j \otimes 1\} \\
&= \{X_i \otimes 1, X_j \otimes 1\} + \{1 \otimes X_i, 1 \otimes X_j\} \\
&+ \{X_i \otimes 1, 1 \otimes X_j\} + \{1 \otimes X_i, X_j \otimes 1\} \\
&= \{X_i, X_j\} \otimes 1 + 1 \otimes \{X_i, X_j\} \\
&= \sum_k \epsilon_{ijk} \left(X_k \otimes 1 + 1 \otimes X_k\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_k \epsilon_{ijk} \Delta(X_k) \\
&= \Delta\left(\{X_i, X_j\}\right)
\end{aligned}
$$

The extension to arbitrary elements results with exactly the same calculation as in the proof of Theorem 3-2.

□

The element t in the definition of the Yangian is the central element of $\mathrm{U}(\mathrm{sl}(2,\mathbb{C}))$ given by the standard generators $\{e, f, h\}$ and considered as an element in $\mathrm{U}(\mathrm{sl}(2,\mathbb{C}))\otimes\mathrm{U}(\mathrm{sl}(2,\mathbb{C}))$. Of course in the present case it looks much more familiar:

PROPOSITION 4-1. *The element*

$$
C = \sum_{i=1}^{3} X_i^2 \tag{64}
$$

is in the center of $(\mathcal{A}, m, \Delta, \{.,.\})$.

Proof. It is again convenient to show the identity first on the generators:

$$
\begin{aligned}
[C, X_j\} &= \sum_{i=1}^{3} \left[X_i^2, X_j\right\} = \sum_{i=1}^{3} [X_i X_i, X_j\} = \sum_{i=1}^{3} X_i\, [X_i, X_j\} + [X_i, X_j\}\, X_i \\
&= \sum_{i,k=1}^{3} \epsilon_{ijk} X_i X_k + \sum_{i,k=1}^{3} \epsilon_{ijk} X_k X_i = \sum_{i,k=1}^{3} \epsilon_{ijk} X_i X_k - \sum_{i,k=1}^{3} \epsilon_{kji} X_k X_i \\
&= \sum_{i,k=1}^{3} \epsilon_{ijk} X_i X_k - \sum_{i,k=1}^{3} \epsilon_{ijk} X_i X_k \\
&= 0
\end{aligned}
$$

The commutator case results immediately. Using the same decomposition of algebra elements as in the proof of Theorem 2-1 and (53) with $Z = C$ and $a_l = X_{i_l}$, the Poisson case results as well.

□

With that, the algebraic setting is complete and it can be started to consider Theorem 3-2:

LEMMA 4-1. *Considering the central element of Proposition 4-1 and an arbitrary partition tree, the realizations of the algebras given by Definition/Proposition 4-2 as $F(\mathcal{P})$ or $O(\mathcal{H})$ together with Theorem 3-2 give rise to a complete set of N commuting operators or N functionally independent functions respectively if $X = X_3$.*

Proof. Let ρ denote the respective realization of the algebra. It follows for the commuting functions or operators

$$\begin{aligned}\left(\otimes_{i=1}^{N}\rho\right)\left(\Delta^{M}(C)\right) &= \sum_{i=1}^{3}\left(\left(\otimes_{i=1}^{N}\rho\right)\left(\Delta^{M}\left(X_i\right)\right)\right)^2 = \sum_{i=1}^{3}\left(\sum_{j\in M}\alpha_j^i\right)^2\\ &= \sum_{i=1}^{3}\left(\alpha_M^i\right)^2 = \vec{\alpha}_M^2,\end{aligned}$$

where $M \in \mathcal{B}$. Therefore the constraint $s_i^2 = \|s_i\|^2 = 1$ immediately shows that C is mapped to a constant in the first realization. The realization as spin operators is irreducible and the image of C is a selfintertwiner. Therefore, due to Schur´s Lemma, it is a scalar multiple of the identity. Hence there are N commuting operators or functions.
The functional independence of the functions is obvious, as the total derivative of X_3´s image has only components resulting from the third spin component.
The quantum case is a little more involved. To proof the completeness of a set of operators, one has to show that their common eigenstates are uniquely labelled by their eigenvalues. For that let $V(m)$ again be the usual irreducible $\mathrm{sl}(2,\mathbb{C})$ module. In such a module obviously one label suffices to mark the eigenstates uniquely. Usually the eigenvalues of the third spin component, denoted by S^3, are used for that. As the tensor product of two such modules, corresponding to a two particle system, decomposes into a direct sum of irreducible modules

$$V(m)\otimes V(n) \cong V(m+n)\oplus V(m+n-2)\oplus\ldots\oplus V(|n-m|),$$

two labels are needed. One which gives the component and one for the states. Of course the second is again the z component of the total spin and the first one is given by the total spin squared, which should be designated by the label S. If one now extends to a tensor product of three irreducible components

$$\begin{aligned}(V(m)\otimes V(n))\otimes V(r) &\cong (V(m+n)\oplus V(m+n-2)\oplus\ldots\oplus V(n-m))\otimes V(r)\\ &\cong V(n+m)\otimes V(r)\oplus V(n+m-2)\otimes V(r)\\ &\oplus \ldots\oplus V(n-m)\otimes V(r),\end{aligned}$$

obviously a third label is necessary to distinguish the components appearing more than one time in the final result of the reduction. As the components appearing several times result from the two particle system, it is clear that the third label

can chosen to be its total spin squared S_1.
This can be continued giving the general statement that if the eigenstates of a first system are uniquely labelled by (S_1, S_1^3, R_1), where R_1 denotes the set of labels given by the total spin squared of former subsystems in the sense of the above considerations, and those of a second system by (S_2, S_2^3, R_2), the states of the complete system are uniquely labelled by $(S, S^3, S_1, S_2, R_1, R_2)$. The above considerations for the two special cases of course immediately show that just "existing" labels are adopted for the tensor case so that the statement is universally valid. It results immediately by a glimpse on the examples:
The tupel S_i, R_i with $i = 1, 2$ distinguishes exactly one irreducible component for the i-th system. Therefore (S_1, S_2, R_1, R_2) distinguishes exactly one tensor product of irreducible components in the tensor case. As seen above, it is a crucial aspect of the Clebsch-Gordon decomposition that in the tensor product of two irreducible modules, every irreducible component appears only once. Therefore the total spin S is a unique label in the respective subsystem and S^3 labels the associated states.
This means that considering (S, S^3) and then following a complete decomposition of the system associated with S_i always gives a complete set of labels. It was shown in the proof of Lemma 4-1 that the commuting operators resulting from Theorem 3-3 are the total spins squared. As for the $\mathcal{B}$-partitioned spin systems the arbitrary algebra element is chosen to be $X = X_3$, giving the z component of the total spin, the resulting labels give exactly this set. Hence the proof is finished.

□

The progress made up to now is summarized in (Figure 4).

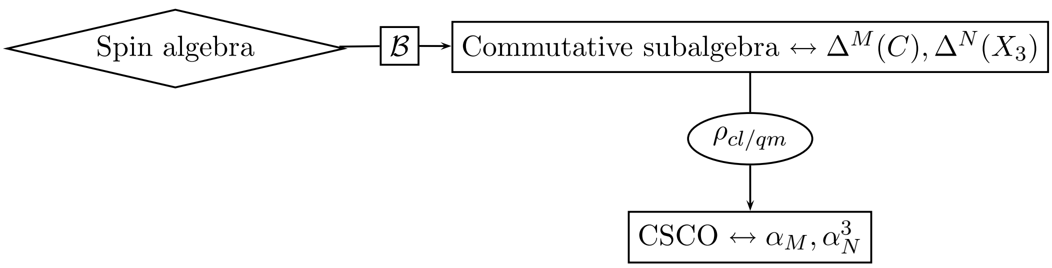

Figure 4: Schematic sketch associated with the use of Theorem 3-2 (Abbreviations: **C**omplete **S**ets of **C**ommuting **O**perators, **cl**assical, **q**uantum **m**echanical)

The second lemma makes a concrete connection to the $\mathcal{B}$-partitioned systems [8]:

LEMMA 4-2. *The Hamiltonian of every $\mathcal{B}$-partioned spin system can be written in terms of the commuting operators or functions respectively, resulting from the corresponding representations of $(\mathcal{A}, m, \Delta, \{.,.\})$.*

Proof. Let $M \in \mathcal{B}$ and again $\vec{\alpha}_M^2 = \left(\sum_{i\in M} \vec{\alpha}_i\right)^2$. The first step is to show that the Hamiltonian of any $\mathcal{B}$-partitioned spin system can be rewritten in terms of α_M^2

$$\begin{aligned}
\frac{1}{2}\sum_{M\in\mathcal{B}} J(M)\left(\vec{\alpha}_M^2 - \vec{\alpha}_{M_1}^2 - \vec{\alpha}_{M_2}^2\right) &= \frac{1}{2}\sum_{M\in\mathcal{B}} J(M)\left((\vec{\alpha}_{M_1} + \vec{\alpha}_{M_2})^2 - \vec{\alpha}_{M_1}^2 - \vec{\alpha}_{M_2}^2\right) \\
&= \sum_{M\in\mathcal{B}} J(M)\vec{\alpha}_{M_1}\vec{\alpha}_{M_2} \\
&= \sum_{M\in\mathcal{B}} J(M)\left(\sum_{i\in M_1}\vec{\alpha}_i\right)\left(\sum_{i\in M_2}\vec{\alpha}_j\right) \\
&= \sum_{M\in\mathcal{B}}\left(\sum_{i\in M_1, j\in M_2} J(M_{ij})\vec{\alpha}_i\vec{\alpha}_j\right) \\
&= \sum_{i,j} J(M_{ij})\vec{\alpha}_i\vec{\alpha}_j,
\end{aligned} \tag{65}$$

where of course $J(\{\mu\}) = 0$. Defining $\lfloor M \rfloor$ to be the smallest set containing M, except for M itself, $\lfloor\{1,\ldots,N\}\rfloor = 0$ and $J(0) = 0$, a glimpse on the first terms of the right hand side of the first line of (65) gives:

$$\begin{aligned}
\frac{1}{2}\sum_{M\in\mathcal{B}} J(M)\left(\vec{\alpha}_M^2 - \vec{\alpha}_{M_1}^2 - \vec{\alpha}_{M_2}^2\right) &= \frac{1}{2}J(M)\left(\vec{\alpha}_M^2 - \vec{\alpha}_{M_1}^2 - \vec{\alpha}_{M_2}^2\right) \\
&+ \frac{1}{2}J(M_1)\left(\vec{\alpha}_{M_1}^2 - \vec{\alpha}_{M_{11}}^2 - \vec{\alpha}_{M_{12}}^2\right) \\
&+ \frac{1}{2}J(M_2)\left(\vec{\alpha}_{M_2}^2 - \vec{\alpha}_{M_{21}}^2 - \vec{\alpha}_{M_{22}}^2\right)\ldots \\
&= \frac{1}{2}J(M)\vec{\alpha}_M^2 + \frac{1}{2}\left(J(M_1) - J(M)\right)\vec{\alpha}_{M_1}^2 \\
&+ \frac{1}{2}\left(J(M_2) - J(M)\right)\vec{\alpha}_{M_2}^2 + \ldots
\end{aligned}$$

As $\lfloor M_1 \rfloor = \lfloor M_2 \rfloor = M$, the following identity results immediately:

$$H = \frac{1}{2}\sum_{M\in\mathcal{B}}\left(J(M) - J\left(\lfloor M\rfloor\right)\right)\vec{\alpha}_M^2 \tag{66}$$

From the proof of Lemma 4-1 it is clear that the commuting elements are exactly $\vec{\alpha}_M^2$ for $M \in \mathcal{B}$. This concludes the proof.

□

Now the crucial theorem can be proven:

THEOREM 4-1. *Every $\mathcal{B}$-partitioned spin system is integrable.*

Proof. Due to Lemma 4-1 and Lemma 4-2, the Hamitonian of every $\mathcal{B}$-partitioned spin system can be considered as a part of a set of N commuting operators or functions. As the functions are functionally independent, again due to Lemma 4-1, the integrability of the classical case results immediately from the Liouville-Arnold theorem.
For the quantum case the ranges of the labels have to be determined and the coefficients of the states associated with an arbitrary basis of the respective $\mathrm{sl}(2,\mathbb{C})$ module have to be calculated. The sets R_1 and R_2 from the proof of Lemma 4-1 determine the range of S_1 and S_2. The label S is then given by

$$|S_1 - S_2| \leq S \leq S_1 + S_2,$$

defining the range of S^3 by

$$-S \leq S^3 \leq S.$$

It should be stressed that this of course applies to every subsystem, so that all the ranges are determined.
The coefficients associated to the usual basis

$$\left\{|S_1^3, \ldots, S_N^3\rangle\right\} \tag{67}$$

are called the Clebsch-Gordon coefficients CG. As these are available from the literature [22], it is convenient to organize the eigenstates corresponding to (67), giving [8]

$$|S, S^3, S_1, S_2, R_1, R_2\rangle = |S_{M\in\mathcal{B}_1}, S_N^3\rangle$$
$$= \sum_{S^3_{M\in\mathcal{B}_N}} \left(\prod_{M\in\mathcal{B}_1} CG\left(S_{M_1}, S_{M_2}, S^3_{M_1}, S^3_{M_2}, S_M, S^3_M\right) \right) |S_1^3, \ldots, S_N^3\rangle,$$

where

$$\begin{aligned} \mathcal{B}_1 &= \{M \in \mathcal{B} \,|\, |M| > 1\} \\ \mathcal{B}_N &= \{M \in \mathcal{B} \,|\, M \neq N\}\,. \end{aligned}$$

□

The interaction of a physical system with a time independent external magnetic field $\vec{B}$ is described by the so-called Zeeman term

$$H_{ZM} = c_{ZM} \vec{B}\vec{S}, \tag{68}$$

where c_{ZM} is constant. Choosing the direction of the magnetic field to be the z direction, it is immediately clear from the above considerations that the $\mathcal{B}$-partitioned spin systems can be generalized to an interaction with a magnetic field, without destroying the integrability:

COROLLARY 4-1. *Every $\mathcal{B}$-partitioned system interacting with a time-independent external magnetic field is integrable.*

The section will now be completed by a simple example, considering the following partition tree:

$$\mathcal{B} = \{\{1,2,3,4,5,6\}, \{1,2,3,4\}, \{1,4\}, \{2,3\}, \{5,6\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\} \tag{69}$$

Choosing $J(\{1,4\}) = J(\{2,3\}) = J(\{5,6\}) = 0$ and $J(\{1,2,3,4,5,6\}) = 1$, this gives the Hamiltonian of an octahedron, where sites of the spins are numbered like in (Figure 5) on the next page. Besides the Hamiltonian there are the following constants of motion:

$$\begin{aligned}
\left(\otimes_{i=1}^{6}\rho\right)\left(\Delta^{\{1,2,3,4\}}(C)\right) &= (\vec{\alpha}_1+\vec{\alpha}_2+\vec{\alpha}_3+\vec{\alpha}_4)^2 \\
&= 4 + 2\left(\vec{\alpha}_1\vec{\alpha}_2+\vec{\alpha}_1\vec{\alpha}_3+\vec{\alpha}_1\vec{\alpha}_4+\vec{\alpha}_2\vec{\alpha}_3+\vec{\alpha}_2\vec{\alpha}_4+\vec{\alpha}_3\vec{\alpha}_4\right) \\
\left(\otimes_{i=1}^{6}\rho\right)\left(\Delta^{\{1,4\}}(C)\right) &= (\vec{\alpha}_1+\vec{\alpha}_4)^2 = 2+2\vec{\alpha}_1\vec{\alpha}_4 \\
\left(\otimes_{i=1}^{6}\rho\right)\left(\Delta^{\{2,3\}}(C)\right) &= (\vec{\alpha}_2+\vec{\alpha}_3)^2 = 2+2\vec{\alpha}_2\vec{\alpha}_3 \\
\left(\otimes_{i=1}^{6}\rho\right)\left(\Delta^{\{5,6\}}(C)\right) &= (\vec{\alpha}_5+\vec{\alpha}_6)^2 = 2+2\vec{\alpha}_5\vec{\alpha}_6 \\
\left(\otimes_{i=1}^{6}\rho\right)\left(\Delta^{\{1,2,3,4,5,6\}}(X_3)\right) &= \sum_{i=1}^{6}\alpha_i^3
\end{aligned}$$

Strictly speaking, the second equalities are only valid in the classical case, due to the constraint $s_i^2 = \|s_i\|^2 = 1$. In the quantum case, the first term has to be multiplied with a complex number, due to Schur´s Lemma. The 64 labels for the $S = 1/2$ quantum case are given in the appendix (Table 1,2).

4.2. XXX-S=1/2-Heisenberg chain

The XXX-S=1/2-Heisenberg chain is described by the following Hamiltonian

$$H = -2J\sum_{i=1}^{N}\vec{S}_i\vec{S}_{i+1}, \tag{70}$$

where $S = 1/2$ operators and periodic boundary conditions are imposed by $N+1 = N$. Graphically this corresponds to a ring with N sites, each of which is occupied by a spin 1/2. This system will first be solved by algebraic methods.

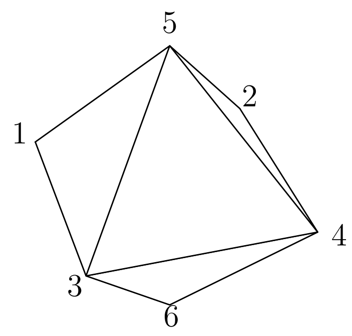

Figure 5: Octahedron with the chosen numbering of the spin sites

4.2.1. Algebraic Bethe ansatz

In the first chapter the Yangian was introduced as one of the most important Quantum Groups. In this section its physical importance will become clear by taking advantage of its (pseudo-)quasicocommutativity in the context of Theorem 3-3.
The spin realization associated with the matrix (35) satisfies the condition of Lemma 3-5. Therefore $T(\lambda)$ is a representation, giving the spin representation of the Yangian, as can be seen from the proof of Theorem 2-1. As the parameter of the spin representation results from the spin realization and not the T matrix, it is clear that, regarded as a representation, $T(\lambda)$ is parameter independent. It gives the spin representation for all values of λ, which is just defined to be same as in the spin realization, as otherwise it would not be possible to define the product structure by means of selfduality.
Due to Theorem 3-3, $T(\lambda)$ satisfies the RTT relations with the Yang matrix (Figure 6). Multiplying both sides with $(-1) = i^2$ shows that

$$iT(\lambda) = \begin{pmatrix} i\lambda + iS^3 & iS^- \\ iS^+ & i\lambda - iS^3 \end{pmatrix} \tag{71}$$

satisfies the RTT relations, too. Passing to a new parameter $i\lambda \to \lambda$ (the Yang matrix (30) is compatible with this scaling) and avoiding the i on the left hand side of (71) in the same sense, leads to the final RTT relations, crucial for the following algebraic Bethe ansatz

$$R^\rho(\mu-\lambda)\left(\begin{pmatrix}\lambda+iS^3 & iS^- \\ iS^+ & \lambda-iS^3\end{pmatrix}\otimes 1\right)\left(1\otimes\begin{pmatrix}\mu+iS^3 & iS^- \\ iS^+ & \mu-iS^3\end{pmatrix}\right)$$
$$=\left(1\otimes\begin{pmatrix}\mu+iS^3 & iS^- \\ iS^+ & \mu-iS^3\end{pmatrix}\right)\left(\begin{pmatrix}\lambda+iS^3 & iS^- \\ iS^+ & \lambda-iS^3\end{pmatrix}\otimes 1\right)R^\rho(\mu-\lambda) \qquad (72)$$

It has to be remembered that so far the RTT relations have corresponded to the product of the spin realization. Hence at this point the question arises, whether it is possible to define the action of the product of RTT algebra elements on the Hilbert space as the usual operator product. Indeed this is possible, because the parameter of the T matrix is defined to be the same as the one of the modules. It is easy to check this by direct computation.

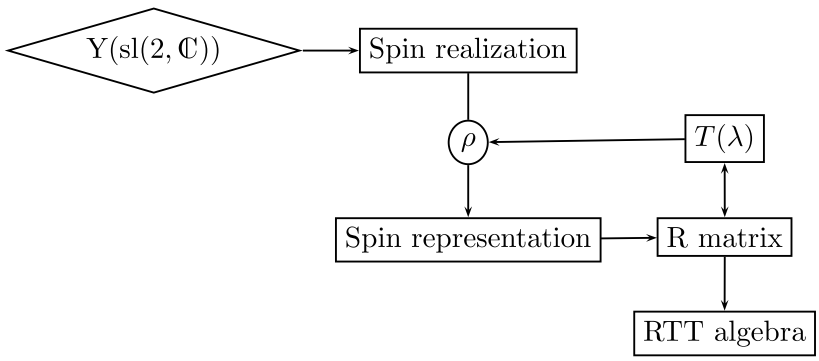

Figure 6: Schematic sketch of the RTT algebra´s construction

Introducing an index n for the respective Hilbert space of the chain and applying Corollary 3-1 shows that the so-called monodromy matrix

$$M(\lambda)=\prod_{n=1}^{N}T_n(\lambda)=\begin{pmatrix}A(\lambda) & B(\lambda)\\ C(\lambda) & D(\lambda)\end{pmatrix} \qquad (73)$$

satisfies the RTT algebra with the same R matrix. This is a key point, because the elements of $M(\lambda)$ are operators on the Hilbert space of the complete chain $\mathcal{H}=\otimes_{n=1}^{N}\mathcal{H}_n$. In accordance with the introduction of the preceding chapter, it now has to be shown that the Hamiltonian of the chain can be written in terms of RTT algebra elements. For that it is helpful to clarify the situation of the spaces involved in the RTT relations. The R matrix acts in the tensor product of $V(1)_\lambda$ and $V(1)_\mu$, where the associated sl$(2,\mathbb{C})$ module $V(1)$ is often called the "**a**uxiliary" space in this context. Accordingly $T_n(\lambda)$ has to be interpreted as an operator on the n-th quantum and the auxiliary space

$$T_n(\lambda)=\lambda 1_n\otimes 1_a+i\sum_{i=1}^{3}S_n^i\otimes\sigma_a^i, \qquad (74)$$

where σ_a^i for $i = 1, 2, 3$ is considered as an abstract operator. Hence choosing the S^3 eigenbasis for the auxiliary space reproduces the T matrix (71). The switch operator between the n-th quantum and the auxiliary space can be written as

$$\tau_{na} = \frac{1}{2}\left(1_n \otimes 1_a + \sum_{i=1}^{3} \sigma_n^i \otimes \sigma_a^i\right) \tag{75}$$

and hence it follows that

$$T_n(\lambda) = \left(\lambda - \frac{i}{2}\right) 1_n \otimes 1_a + i\tau_{na}. \tag{76}$$

This means that at $\lambda = i/2$ the T matrix becomes the switch operator between both of the spaces:

$$T_n(i/2) = i\tau_{na} \tag{77}$$

It will become clear within the proof of the following proposition, mainly adopted from [3], that this is crucial, because the Hamiltonian can be considered as a superposition of switch operators:

PROPOSITION 4-2. *Let* $\mathrm{Tr}_a(\ldots)$ *denote the trace over the auxiliary space. The Hamiltonian (70) can be written as:*

$$H = -iJ\frac{d}{d\lambda}\ln\left(\mathrm{Tr}_a\left(M(\lambda)\right)\right)|_{\lambda=i/2} + J\frac{N}{2} \tag{78}$$

Proof. The first step is to calculate the trace of $M(i/2)$

$$\begin{aligned}
\mathrm{Tr}_a\left(M(i/2)\right) &= i^N\,\mathrm{Tr}_a\left(\tau_{1a}\ldots\tau_{Na}\right) \\
&= i^N\,\mathrm{Tr}_a\left(\tau_{(N-1)N}\tau_{(N-2)(N-1)}\ldots\tau_{12}\tau_{1a}\right) \\
&= i^N\tau_{(N-1)N}\tau_{(N-2)(N-1)}\ldots\tau_{12}\,\mathrm{Tr}_a\left(\tau_{1a}\right) \\
&= i^N\tau_{(N-1)N}\tau_{(N-2)(N-1)}\ldots\tau_{12},
\end{aligned} \tag{79}$$

which is, up to multiplication with i^N, nothing but a complete shift

$$|\alpha_1\ldots\alpha_N\rangle \overset{i^{-N}\,\mathrm{Tr}_a(M(\lambda))}{\rightarrow} |\alpha_2\ldots\alpha_N\alpha_1\rangle,$$

where a general state in in $\mathcal{H}$ has been considered. Because of

$$\frac{d}{d\lambda}T_n(\lambda) = 1_n \otimes 1_a,$$

it follows

$$\begin{aligned}
\frac{d}{d\lambda}M(\lambda)|_{\lambda=i/2} &= i^{N-1}\sum_{j=1}^{N} T_1(i/2)\ldots\hat{T}_j(i/2)\ldots T_n(i/2) \\
&= i^{N-1}\sum_{j=1}^{N} \tau_{1a}\ldots\tau_{(j-1)a}\hat{\tau}_{ja}\tau_{(j+1)a}\ldots\tau_{Na},
\end{aligned}$$

where the hat means that the respective term is omitted. Tracing out the above result and bearing in mind that the trace and the derivative of course can be interchanged it follows:

$$\frac{d}{d\lambda}\operatorname{Tr}_a\left(M(\lambda)\right)|_{\lambda=i/2} = i^{N-1}\sum_{j=1}^{N} \underbrace{\tau_{(N-1)N}\tau_{(N-2)(N-1)}\cdots}_{(*)}\tau_{(j-1)(j+1)}\underbrace{\tau_{(j+1)(j+2)}\cdots\tau_{12}}_{(**)} \tag{80}$$

Remembering that (79) gives a shift, the application of (80) to an arbitrary state in $\mathcal{H}$ gives

$$\begin{aligned}
|\alpha_1\ldots\alpha_N\rangle &\overset{(**)}{\rightarrow} |\alpha_2\ldots\alpha_{(j-1)}\alpha_1\alpha_{(j)}\alpha_{(j+1)}\alpha_{(j+2)}\ldots\alpha_N\rangle \\
&\overset{\tau_{(j-1)(j+1)}}{\rightarrow} |\alpha_2\ldots\alpha_{(j-1)}\alpha_{(j+1)}\alpha_j\alpha_{(1)}\alpha_{(j+2)}\ldots\alpha_N\rangle \\
&\overset{(*)}{\rightarrow} |\alpha_2\ldots\alpha_{(j-1)}\alpha_{(j+1)}\alpha_{(j)}\alpha_{(j+2)}\alpha_{(j+3)}\ldots\alpha_N\alpha_1\rangle.
\end{aligned}$$

Hence (80) is just a complete shift and a switch. According to (79) the complete shift can be written as the trace of the monodromy matrix over the auxiliary space at $\lambda = i/2$ and hence it follows:

$$\frac{d}{d\lambda}\operatorname{Tr}_a\left(M(\lambda)\right)|_{\lambda=i/2} = i^{-1}\operatorname{Tr}_a\left(M(i/2)\right)\sum_{j=1}^{N}\tau_{j(j+1)}$$

It results immediately from the identity

$$\vec{S}_i\vec{S}_{i+1} = \frac{1}{2}\left(S_i^+S_{i+1}^- + S_i^-S_{i+1}^+\right) + S_i^3S_{i+1}^3$$

that

$$\vec{S}_i\vec{S}_{i+1} = \frac{1}{2}\tau_{i(i+1)} - \frac{1}{4}.$$

Therefore it results

$$\begin{aligned}
H &= -iJ\left(\operatorname{Tr}_a M(i/2)\right)^{-1}\frac{d}{d\lambda}\operatorname{Tr}_a\left(M(\lambda)\right)|_{\lambda=i/2} + J\frac{N}{2} \\
&= -iJ\frac{d}{d\lambda}\ln\left(\operatorname{Tr}_a M(\lambda)\right)|_{\lambda=i/2} + J\frac{N}{2},
\end{aligned}$$

which completes the proof.

□

Now one can start to work for the algebraic Bethe ansatz, where the slightly more general problem of diagonalizing the trace of the monodromy matrix for arbitrary λ will be considered. The correct eigenvalues for the XXX-S=1/2-Heisenberg Hamiltonian then result by setting $\lambda = i/2$. To start with the algebraic Bethe ansatz, an eigenstate of $A(\lambda)+D(\lambda)$, relative to which elements of the RTT algebra can be interpreted as creation or annihilation operators, has to be found. The complete spin-up state

$$|\Omega\rangle = \otimes_{i=1}^{N}|1\rangle \tag{81}$$

is of course an eigenstate of $\mathrm{Tr}_a(M(\lambda))$

$$\mathrm{Tr}_a(M(\lambda))|\Omega\rangle = (A(\lambda)+D(\lambda))\,|\Omega\rangle = (\xi(\lambda)+\zeta(\lambda))\,|\Omega\rangle, \tag{82}$$

where $\xi(\lambda) = (\lambda + i/2)^N$ and $\zeta(\lambda) = (\lambda - i/2)^N$. A look at (71) shows that it is annihilated by $C(\lambda)$. Therefore it is natural to ask $B(\lambda)$ to be a creation operator. This means that λ has to be chosen in a way that

$$|\Omega_1\rangle := B(\lambda_1)|\Omega\rangle \tag{83}$$

or more general

$$|\Omega_r\rangle := \prod_{i=1}^{r} B(\lambda_i)|\Omega\rangle \tag{84}$$

is an eigenstate of $\mathrm{Tr}_a(M(\lambda))$, too.
As for that $\mathrm{Tr}_a(M(\lambda))B(\lambda_i)|\Omega\rangle$ has to be calculated, it is necessary to derive the commutation relations of $A(\lambda)$ and $B(\mu)$, $B(\lambda)$ and $B(\mu)$ and $D(\lambda)$ and $B(\mu)$. These result from the RTT relations [23]:

PROPOSITION 4-3. *The elements $A(\lambda)$, $B(\lambda)$ and $D(\lambda)$ satisfy the following commutation relations*

$$B(\lambda)A(\mu)c(\mu-\lambda) + A(\lambda)B(\mu)b(\mu-\lambda) = a(\mu-\lambda)B(\mu)A(\lambda) \tag{85}$$

$$[B(\lambda), B(\mu)] = 0 \tag{86}$$

$$B(\lambda)D(\mu)a(\mu-\lambda) = c(\mu-\lambda)B(\mu)D(\lambda) + b(\mu-\lambda)D(\mu)B(\lambda), \tag{87}$$

where $a(\lambda) = 1$, $b(\lambda) = \frac{\lambda}{\lambda-i}$ and $c(\lambda) = -\frac{i}{\lambda-i}$ are the elements of the R matrix.

Proof. Writing down the RTT relations explicitely, again in terms of the product structure on the tensor product, gives:

$$\sum_{k_3 l_3} M(\lambda)_{k_1 k_3} M(\mu)_{l_1 l_3} R(\mu-\lambda)_{k_3 l_2}^{l_3 k_2} = \sum_{k_3 l_3} R(\mu-\lambda)_{k_1 l_3}^{l_1 k_3} M(\mu)_{k_3 k_2} M(\lambda)_{l_3 l_2}$$

Choosing $k_1 = 1$, $l_1 = 1$, k_2=2 and l_2 =1 leads to:

$$\begin{aligned} M(\lambda)_{12}M(\mu)_{11}R(\mu-\lambda)_{21}^{12} \;&+\; M(\lambda)_{11}M(\mu)_{12}R(\mu-\lambda)_{11}^{22} \\ &= R(\mu-\lambda)_{11}^{11}M(\mu)_{12}M(\lambda)_{11} \end{aligned} \tag{88}$$

All terms containing elements of $R(\mu-\lambda)$ which are equal to zero have been omitted. The relation (88) gives the first of the three commutation relations (85):

$$B(\lambda)A(\mu)c(\mu-\lambda)+A(\lambda)B(\mu)b(\mu-\lambda)=a(\mu-\lambda)B(\mu)A(\lambda)$$

The second and the third are derived within equivalent calculations with $k_1=1$, $l_1=1$, k_2=2 and l_2 =2 for (86) and $k_1=1$, $l_1=2$, k_2=2 and l_2 =2 for (87).

□

Now the creation operator can be constructed or equivalentely, the Bethe ansatz equations can be obtained. In this context it is extremely helpful to examine an easy case first. Using (85) and (87) for the r=1 case the following relation results [23]:

$$\begin{aligned}\mathrm{Tr}_a(M(\lambda))|\Omega_1\rangle &= \left[\xi(\lambda)\frac{a(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}+\zeta(\lambda)\frac{a(\lambda-\lambda_1)}{b(\lambda-\lambda_1)}\right]|\Omega_1\rangle \\ &- \left[\xi(\lambda_1)\frac{c(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}+\zeta(\lambda_1)\frac{c(\lambda-\lambda_1)}{b(\lambda-\lambda_1)}\right]B(\lambda)|\Omega\rangle \qquad (89)\end{aligned}$$

Apparently $|\Omega_1\rangle$ is an eigenstate of $\mathrm{Tr}_a(M(\lambda))$ if the so-called "unwanted terms" in the second bracket vanish. This postulate gives the Bethe ansatz equations for the r=1 case. The eigenvalue is then given by the terms in the first bracket.
The results of the general case will be summarized in the following proposition:

PROPOSITION 4-4. *The eigenstates of* $\mathrm{Tr}_a\left(M(\lambda)\right)$ *are given by*

$$\prod_{i=1}^{r} B\left(\lambda_i\right)|\Omega\rangle, \qquad (90)$$

where the parameters λ_i *are the so-called Bethe ansatz roots, which have to satisfy the Bethe ansatz equations*

$$\left(\frac{\lambda_j+i/2}{\lambda_j-i/2}\right)^N=\prod_{i=1,i\neq j}^{r}\frac{\lambda_j-\lambda_i+i}{\lambda_j-\lambda_i-i}. \qquad (91)$$

The eigenvalues $E(\lambda)$ *are given by:*

$$E(\lambda)=\xi(\lambda)\prod_{i=1}^{r}\frac{a(\lambda_i-\lambda)}{b(\lambda_i-\lambda)}+\zeta(\lambda)\prod_{i=1}^{r}\frac{a(\lambda-\lambda_i)}{b(\lambda-\lambda_i)} \qquad (92)$$

Proof. The proof of the general case essentially copies the computation of the special one, presented above. Therefore it starts at the following point:

$$\mathrm{Tr}_a(M(\lambda))|\Omega_r\rangle = A(\lambda)B(\lambda_1)\prod_{i=2}^{r} B(\lambda_i)|\Omega\rangle + D(\lambda)B(\lambda_1)\prod_{i=2}^{r} B(\lambda_i)|\Omega\rangle \tag{93}$$

Due to the formal similarity between both of the above terms, it suffices to analyze one of them explicitely. Using (85) the first term gives:

$$\begin{aligned} A(\lambda)B(\lambda_1)\prod_{i=2}^{r} B(\lambda_i)|\Omega\rangle &= \underbrace{\frac{a(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}B(\lambda_1)A(\lambda)}_{(*)}\prod_{i=2}^{r} B(\lambda_i)|\Omega\rangle \\ &- \underbrace{\frac{c(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}B(\lambda)A(\lambda_1)}_{(**)}\prod_{i=2}^{r} B(\lambda_i)|\Omega\rangle \end{aligned} \tag{94}$$

It results immediately from (89) that the eigenvalue $E(\lambda)$ must be given by the terms with $\xi(\lambda)$ and $\zeta(\lambda)$. As one can get from (*) in (94), these result from the product of the first terms within the successive use of (85). Therefore it is given by

$$E(\lambda) = \xi(\lambda)\prod_{i=1}^{r}\frac{a(\lambda_i-\lambda)}{b(\lambda_i-\lambda)} + \zeta(\lambda)\prod_{i=1}^{r}\frac{a(\lambda-\lambda_i)}{b(\lambda-\lambda_i)}. \tag{95}$$

Now the unwanted terms have to be derived, which are given by the terms with $\xi(\lambda_i)$ and $\zeta(\lambda_i)$, where $i = 1, 2, \dots, r$. As all these result from the successive use of (85) and are easy to generalize therefore, it suffices to derive the relations for i=1.

Since in the further use of (85) only terms with $i > 1$ result from the first part of (94), only the second one has to be examined. With the same argumentation as above and (86), which is necessary to combine the appearing terms differing by the order of the creation operators, one immediately gets from (**) in (94) the part of the unwanted terms generated by $A(\lambda_1)$. Adding the part resulting from $D(\lambda_1)$ gives:

$$\xi(\lambda_1)\frac{c(\lambda_1-\lambda)}{b(\lambda_1-\lambda)}\prod_{i=2}^{N}\frac{a(\lambda_i-\lambda_1)}{b(\lambda_i-\lambda_1)} + \zeta(\lambda_1)\frac{c(\lambda-\lambda_1)}{b(\lambda-\lambda_1)}\prod_{i=2}^{N}\frac{a(\lambda_1-\lambda_i)}{b(\lambda_1-\lambda_i)} \tag{96}$$

Postulating (96) to vanish and taking the values of $a(\lambda)$, $b(\lambda)$ and $c(\lambda)$ into account, it follows that

$$\frac{\xi(\lambda_1)}{\zeta(\lambda_1)} = \prod_{i=2}^{r}\frac{b\,(\lambda_1-\lambda_i)}{b(\lambda_i-\lambda_1)},$$

which generalizes to

$$\frac{\xi(\lambda_j)}{\zeta(\lambda_j)} = \prod_{i=1, i\neq j}^{r} \frac{b\,(\lambda_j - \lambda_i)}{b\,(\lambda_i - \lambda_j)},$$

where $j = 1, 2, \ldots, r$.

This reads explicitely

$$\left(\frac{\lambda_j + i/2}{\lambda_j - i/2}\right)^N = \prod_{i=1, i\neq j}^{r} \frac{\lambda_j - \lambda_i + i}{\lambda_j - \lambda_i - i},$$

completning the proof.

□

4.2.2. Coordinate Bethe ansatz

Now the original calculation done by Bethe and Hulthén will be presented, mainly following the considerations made by Yosida [24]. In contrast to the algebraic Bethe ansatz this is an analytical approach for the construction of creation and annihilation operators. Although a few current problems have been solved this way, it is more or less completely replaced by the algebraic solution, which is much easier to handle in many situations.
The Hamiltonian of the chain acts on $\mathcal{H}$, which is spanned by the orthonormal basis resulting from the eigenstates of S^3

$$\mathcal{H} = \langle\{|\alpha_j\rangle\}\rangle = \left\langle\left\{\otimes_{i=1}^{N} |\alpha_{ij}\rangle\right\}\right\rangle, \tag{97}$$

where $|\alpha_{ij}\rangle$ can either be the spin-up or the spin-down state and j goes from 1 to $\dim(\mathcal{H})$=2^N.
If $|\alpha\rangle$ denotes an eigenstate of H, it must be given by a linear combination of the basis elements:

$$|\alpha\rangle = \sum_{j=1}^{2^N} a_j |\alpha_j\rangle \tag{98}$$

The aim is to find the correct components a_j. It is extremely helpful to split up the problem into the search on different subspaces associated with different numbers of spin down states, where $|\Omega\rangle$ again works as the reference state. These subspaces are usually called magnon sectors. An r-magnon state $|\alpha_r\rangle$ is given by

$$|\alpha_r\rangle = \sum_{\mathbf{m}} a(\mathbf{m})|\alpha(\mathbf{m})\rangle := B|\Omega\rangle, \tag{99}$$

where $\mathbf{m}$ is the multi index

$$1 \leq m_1 < m_2 \ldots < m_{r-1} < m_2 \leq N \tag{100}$$

and B is defined by:

$$B = \sum_{\mathbf{m}} a(\mathbf{m}) \sigma^-_{m_1} \dots \sigma^-_{m_r} \tag{101}$$

Now an equation for the coordinates $a(\mathbf{m})$ has to be derived. For that the matrix of the Hamiltonian associated with the basis of the r-magnon sector has to be calculated. The diagonal elements are given by

$$H_{\mathbf{mm}} = J\left(\frac{N}{2} - N_p\right), \tag{102}$$

where N_p denotes the number of parallel spin pairs which are nearest neighbours. Off-diagonal elements occur whenever the respective bra- and ket-state differ by an inverted anti-parallel, nearest neighbour spin pair. Therefore the number of off-diagonal entries in a row is given by $(N - N_p)$, where N_p of course refers to the bra-state. The off-diagonal elements read:

$$H_{\mathbf{mn}} = -J \tag{103}$$

The relations (102) and (103) can now be used to single out one basis element in the Schrödinger equation:

$$\begin{aligned}
\langle \alpha(\mathbf{n})|H|\alpha_r\rangle &= \sum_{\mathbf{m}} a(\mathbf{m}) \langle \alpha(\mathbf{n})|H|\alpha(\mathbf{m})\rangle \\
&= a(\mathbf{n})\langle \alpha(\mathbf{n})|H|\alpha(\mathbf{n})\rangle + \sum_{\mathbf{m},\mathbf{m}\neq\mathbf{n}} a(\mathbf{m})\langle \alpha(\mathbf{n})|H|\alpha(\mathbf{m})\rangle \\
&= Ja(\mathbf{n})\left(\frac{N}{2} - N_p\right) - J\sum_{\mathbf{q}} a(\mathbf{q}) \\
&= Ea(\mathbf{n})
\end{aligned} \tag{104}$$

The multi index $\mathbf{q}$ denotes the non-zero off-diagonal elements. The last two lines in (104) determe the coordinates

$$\begin{aligned}
a(\mathbf{n})\left(N_p - \frac{N}{2} + \frac{E}{J}\right) + \sum_{\mathbf{q}} a(\mathbf{q}) &= a(\mathbf{n})\left(\frac{E}{J} + \frac{N}{2}\right) - (N - N_p)a(\mathbf{n}) + \sum_{\mathbf{q}} a(\mathbf{q}) \\
&= 2\epsilon a(\mathbf{n}) + \sum_{\mathbf{q}} [a(\mathbf{q}) - a(\mathbf{n})] \\
&= 0,
\end{aligned} \tag{105}$$

where ϵ is defined by

$$2\epsilon = \frac{E}{J} + \frac{N}{2}. \tag{106}$$

The configurations $\mathbf{m}$ appearing in the sum (99) are always ordered due to (100). As only the coordinates related to such configurations have to be determined, it can be assumed that $\mathbf{n}$ always satisfies this ordering. As, according to the above explanations, non-zero off-diagonal entries of the Hamiltonian appear whenever the bra- and ket-state differ by an anti-parallel spin pair, the ordering is violated if spin-up states are located at nearest neighbour sites, giving configurations with equal components. Whenever $m_r = N$, the range of the components is crossed and the ordering breaks, too.

These violations can be taken into consideration by making two additional postulates on the intuitive approach to consider $a(\mathbf{m})$ as a function on the the r-fold tensorial product of the integers, constrained by the ordering. The first one of course has to ensure that the coordinates are functions on the cylic group in the variable which crosses the range defined by the ordering. It will show that it is convenient to postulate this extension to keep the ordering. As S=1/2 is considered, the other violation gives nonsense, because the associated states do not exist. Hence the corresponding terms in the equation have to vanish. Instead of postulating them to be zero, a non-intuitive way will be chosen.

In their most general form at least one of these constraints, as well as the ansatz for the solution itself, is rather cumbersome. Therefore, analogous to the case of the algebraic Bethe ansatz, it is convenient to consider examples first. As aspects of the ordering will play a role, the one and two magnon case should be analysed. Writing down the indices explicitly for r=1, (105) reads:

$$2\epsilon a(m_1) + a(m_1 + 1) + a(m_1 - 1) - 2a(m_1) = 0, \tag{107}$$

This equation is solved by

$$a(m_1) = e^{if_1 m_1}, \tag{108}$$

if

$$\epsilon = 1 - \cos(f_1). \tag{109}$$

If $m_1 = r$, the argument goes beyond the range. Continuing $a(m_1)$ as a function on the cyclic group

$$a(m_1 + N) = a(m_1), \tag{110}$$

gives

$$f_1 = \frac{2\pi}{N}\kappa, \tag{111}$$

with $\kappa = 0, 1, \ldots, (N-1)$. If this so-called Bethe quantum number is chosen to be greater than $(N-1)$, original solutions are reproduced. In the two magnon case the equation (105) reads

$$\begin{aligned} -2\epsilon a(m_1, m_2) &= a(m_1 + 1, m_2) + a(m_1 - 1, m_2) \\ &+ a(m_1, m_2 + 1) + a(m_1, m_2 - 1) - 4a(m_1, m_2). \end{aligned} \tag{112}$$

If the two spins are on nearest neighbour sites, this becomes

$$\begin{aligned} -2\epsilon a(m_1, m_1 + 1) &= a(m_1 + 1, m_1 + 1) + a(m_1 - 1, m_1 + 1) \\ &+ a(m_1, m_1 + 2) + a(m_1, m_1) - 4a(m_1, m_1 + 1), \end{aligned} \tag{113}$$

illustrating for the first time the violation due to non-existing states. One proper and somewhat surprising correction is given by the following restriction:

$$a(m_1, m_1) + a(m_1 + 1, m_1 + 1) - 2a(m_1, m_1 + 1) = 0 \tag{114}$$

Extending the second argument of the coordinates to the cyclic group gives:

$$a(m_2, m_1 + N) = a(m_1, m_2) \tag{115}$$

Now the above solution has to be generalized to the r=2 case. This is done by

$$a(m_1, m_2) = c_1 e^{i(f_1 m_1 + f_2 m_2)} + c_2 e^{i(f_2 m_1 + f_1 m_2)}. \tag{116}$$

Inserting this into (112) shows that it is solved if ϵ is given by

$$\epsilon = \sum_{i=1}^{2} \left[1 - \cos(f_i)\right]. \tag{117}$$

Postulating c_1 and c_2 to be complex numbers of modulus one with phases $\phi_{12} = -\phi_{21}$, (114) gives

$$2\cot\left(\frac{\phi}{2}\right) = \cot\left(\frac{f_1}{2}\right) - \cot\left(\frac{f_2}{2}\right), \tag{118}$$

where $|\phi_{12}| = |\phi_{21}| = \phi/2$ has been defined. This is the only possibility to gain a set of equations with the correct number of undeterminates. Finally f_1 and f_2 have to be calculated by (115):

$$\begin{aligned} e^{i\left(f_1 m_1 + f_2 m_2 + \frac{1}{2}\right)\phi} + e^{i\left(f_2 m_1 + f_1 m_2 - \frac{1}{2}\phi\right)} &= e^{i\left(f_1 m_2 + f_2 (m_1 + N) + \frac{1}{2}\phi\right)} + e^{i\left(f_2 m_2 + f_1 (m_1 + N) - \frac{1}{2}\phi\right)} \Leftrightarrow \\ e^{i(f_1 m_1 + f_2 m_2)}\left(e^{\frac{i}{2}\phi} - e^{i\left(f_1 N - \frac{1}{2}\phi\right)}\right) + e^{i(f_2 m_1 + f_1 m_2)}\left(e^{-\frac{i}{2}\phi} - e^{i\left(f_2 N + \frac{1}{2}\phi\right)}\right) &= 0 \end{aligned} \tag{119}$$

As the last equation should hold for all configurations, it is clear that the both terms independent of (m_1, m_2) have to vanish, giving

$$\begin{aligned} N f_1 - \phi &= 2\pi\kappa_1 \\ N f_2 + \phi &= 2\pi\kappa_2, \end{aligned} \tag{120}$$

where κ_1,$\kappa_2 = 0, 1, 2, \ldots, N-2, N-1$.
Now the results of the general case, summarized in the following proposition, can be obtained:

PROPOSITION 4-5. *The eigenstates of the XXX-S=1/2-Heisenberg Hamiltonian are given by*

$$B_r|\Omega\rangle, \tag{121}$$

where the creation operators B_r are given by

$$B_r = \sum_{m_1,\ldots,m_r} a\left(\mathbf{m}\right) \sigma^-_{m_1} \ldots \sigma^-_{m_r}. \tag{122}$$

The coefficients $a\left(\boldsymbol{m}\right)$ are given by

$$a(m_1,\ldots,m_r) = \sum_P e^{i\sum\limits_{i=1}^{r} f_{P_i} m_i + \frac{i}{2}\sum\limits_{j<l}\phi_{P_j P_l}}, \tag{123}$$

where the parameters f_i and ϕ_{ij} are called Bethe roots. They have to satisfy the so-called Bethe equations

$$2\cot\left(\frac{\phi_{kl}}{2}\right) = \cot\left(\frac{f_k}{2}\right) - \cot\left(\frac{f_l}{2}\right) \tag{124}$$

$$N f_k - \sum_{l\neq k}\phi_{kl} = 2\pi\kappa_k, \tag{125}$$

where $\kappa_k = 1, 2, \ldots, N-2, N-1$. Finally the shifted eigenvalues ϵ are given by

$$\epsilon = \sum_{i=1}^{r}\left[1 - \cos\left(f_i\right)\right]. \tag{126}$$

Proof. Writing down (105) analogous to the above two cases gives:

$$\begin{aligned} -2\epsilon a(m_1,\ldots,m_r) &= \sum_{i=1}^{r}[a(m_1,\ldots,m_i+1,\ldots,m_r) \\ &+ a(m_1,\ldots,m_i-1,\ldots,m_r) - 2a(m_1,\ldots m_r)] \end{aligned}$$

The problems related to the S =1/2 situation occur whenever at least two spin-up states are located at nearest neighbour sites. Thus (114) now has to be imposed on every nearest neighbour pair of spins

$$\begin{aligned} a(m_1,\ldots,m_k,m_k,\ldots,m_r) &+ a(m_1,\ldots,m_k+1,m_k+1,\ldots,m_r) \\ &- 2a(m_1,\ldots,m_k,m_k+1,\ldots,m_r) = 0 \end{aligned} \tag{127}$$

with $k = 1,\ldots,N$.

The ansatz (123) for the r-magnon case results immediately from (116), where the summation outside the exponent has only symbolic character, as it denotes a definite sequence of the $r!$ permutations of $\{1,2,\ldots,(r-1),r\}$. The shifted energy ϵ is given by (117), where the sum is taken up to r.

To derive the generalization of (118), exactly like above, (123) has to be inserted in (127). For that it is useful to rewrite (123): To every permutation over which the sum is taken, a counterpart with the k-th and $(k+1)$-th element reversed exists. Thus the sum in (123) can be rewritten over only half the permutations. If this is denoted by P^*, (123) reads

$$a(m_1,\ldots,m_r)=\sum_{P^*}\cdots\left(e^{if_{P_k^*}m_k+if_{P_{k+1}^*}m_{k+1}+\frac{i}{2}\phi_{P_k^*P_{k+1}^*}}+e^{if_{P_{k+1}^*}m_k+if_{P_k^*}m_{k+1}-\frac{i}{2}\phi_{P_k^*P_{k+1}^*}}\right),$$

where the identic terms of the both permutations are omitted for clarity. With this (127) gives:

$$\begin{aligned}
2&\sum_{P^*}\cdots\left(e^{i(f_{P_k^*}+f_{P_{k+1}^*})m_k+if_{P_{k+1}^*}+\frac{i}{2}\phi_{P_k^*P_{k+1}^*}}+e^{i(f_{P_k^*}+f_{P_{k+1}^*})m_k+if_{P_k^*}-\frac{i}{2}\phi_{P_k^*P_{k+1}^*}}\right)\\
=&\sum_{P^*}\cdots\left(e^{i(f_{P_k^*}+f_{P_{k+1}^*})m_k+\frac{i}{2}\phi_{P_k^*P_{k+1}^*}}+e^{i(f_{P_k^*}+f_{P_{k+1}^*})m_k-\frac{i}{2}\phi_{P_k^*P_{k+1}^*}}\right)\\
+&\sum_{P^*}\cdots e^{i(f_{P_k^*}+f_{P_{k+1}^*})m_k+i(f_{P_k^*}+f_{P_{k+1}^*})+if_{P_{k+1}^*}+\frac{i}{2}\phi_{P_k^*P_{k+1}^*}}\\
+&\sum_{P^*}\cdots e^{i(f_{P_k^*}+f_{P_{k+1}^*})m_k+i(f_{P_k^*}+f_{P_{k+1}^*})-\frac{i}{2}\phi_{P_k^*P_{k+1}^*}}
\end{aligned}\tag{128}$$

As each of the sums in (128) is taken over the same set of permutations, they can be combined to one sum. Generally the sum cannot vanish identically so that each addend has to be equal to zero:

$$e^{\frac{i}{2}\phi_{P_k^*P_{k+1}^*}}\left[2e^{if_{P_{k+1}^*}}-1-e^{i(f_{P_k^*}+f_{P_{k+1}^*})}\right]+e^{-\frac{i}{2}\phi_{P_k^*P_{k+1}^*}}\left[2e^{if_{P_k^*}}-1-e^{i(f_{P_k^*}+f_{P_{k+1}^*})}\right]=0\tag{129}$$

This gives

$$2\cot\left(\frac{\phi_{kl}}{2}\right)=\cot\left(\frac{f_k}{2}\right)-\cot\left(\frac{f_l}{2}\right),\tag{130}$$

where the elements P_k^* and P_{k+1}^*, as they take every value within the sum, are replaced by k and l.

Finally the periodic boundary condition has to be generalized. As it results a lot of clarity, it is advisable to reorder the terms in the sum over the permutations in a way that the terms with the same dependence of the configuration appear at the same position:

$$\sum_P e^{i\sum\limits_{i=1}^{r}f_{P_i}m_i+\frac{i}{2}\sum\limits_{j<l}\phi_{P_jP_l}}=\sum_P e^{i\sum\limits_{i=2}^{r}f_{P_i}m_i+f_{P_1}(m_1+N)+\frac{i}{2}\sum\limits_{j<l}\phi_{P_{j+1}P_{l+1}}}$$

As the sums are well-ordered, corresponding to the calculations of the two magnon case, combining to one sum immediately gives

$$Nf_{P_1}+\frac{1}{2}\sum_{k<l}\phi_{P_{k+1}P_{l+1}}-\frac{1}{2}\sum_{k<l}\phi_{P_kP_l}=2\pi\kappa,\tag{131}$$

where $\kappa = 1, \ldots, (N-1)$ due to the same reason as above. Extracting the highest and the lowest terms of the two sums gives

$$
\begin{aligned}
Nf_{P_1} + \frac{1}{2} \sum_{k<l<(r-1)} \phi_{P_{k+1}P_{l+1}} + \frac{1}{2} \sum_{k=1}^{r-1} \phi_{P_{k+1}P_1} - \frac{1}{2} \sum_{2 \leq k < l} \phi_{P_k P_l} - \frac{1}{2} \sum_{l=2}^{r} \phi_{P_1 P_l} \\
= Nf_{P_1} - \frac{1}{2} \sum_{k=1}^{r-1} \phi_{P_1 P_{k+1}} - \frac{1}{2} \sum_{2 \leq k < l} \phi_{P_k P_l} \\
= Nf_{P_1} - \sum_{k=2}^{r} \phi_{P_1 P_k},
\end{aligned}
$$

where the relation $\phi_{ij} = -\phi_{ji}$ has been used. Hence the last relation for the general situation results as

$$
Nf_k - \sum_{l \neq k} \phi_{kl} = 2\pi\kappa_k
$$

with $\kappa_k = 1, \ldots, N-1$.

□

4.2.3. Algebraic and coordinate Bethe ansatz in comparison

In this section the relation between the coordinate and the algebraic Bethe ansatz will be examined. Within the last one the eigenstates are created by the successive use of the same creation or annihilation operators with varying spectral parameter. This is not the case in the original situation, where every state is constructed by an individual operator, which cannot be brought into a form analogous to the one resulting from the algebraic Bethe ansatz. Hence the only way to compare both of the approaches is to make a comparison of the resulting Bethe ansatz equations.
Resolving (129) for ϕ_{ij} and multiplying all terms with different j gives:

$$
\begin{aligned}
\prod_{j=1, j\neq i}^{r} e^{i\phi_{ij}} &= e^{\sum_{j=1, j\neq i}^{r} \phi_{ij}} \\
&= \prod_{j=1, j\neq i}^{r} -\frac{e^{i(f_i+f_j)} + 1 - 2e^{if_i}}{e^{i(f_i+f_j)} + 1 - 2e^{if_j}}
\end{aligned}
\tag{132}
$$

The relation (125) reads:

$$
\sum_{j=1, j\neq i}^{r} \phi_{ij} = Nf_i - 2\pi\kappa_i \tag{133}
$$

Therefore (132) can be rewritten:

$$\left(e^{if_i}\right)^N = \prod_{j=1,j\neq i}^{r} -\frac{e^{i(f_i+f_j)}+1-2e^{if_i}}{e^{i(f_i+f_j)}+1-2e^{if_j}} \tag{134}$$

Introducing

$$\lambda = \frac{i}{2} - \mu, \tag{135}$$

the Bethe ansatz equations read:

$$\frac{\lambda_j - i}{\lambda_j} = \prod_{i=1,i\neq j}^{r} \frac{\lambda_j - \lambda_i - i}{\lambda_j - \lambda_i + i} \tag{136}$$

By comparing the left of (134) with the one of (136) one realizes that the both equations can only be related via the simple transformation

$$e^{if_i} = \frac{\lambda_i - i}{\lambda_i} \tag{137}$$

for all f_i. Insertion of this transformation shows that it is wrong, as it does not reproduce the equations resulting from the algebraic Bethe ansatz. This is crucial, because whenever the R and the T matrix, in its representation in the usual spin basis, are indentical, like for example in [23], the corresponding transformations works. This means that every algebraic Bethe ansatz, resulting from different R and T matrices, is not equivalent to a coordinate Bethe ansatz.
To complete this section, the Bethe ansatz equations derived from both of the approaches will be solved for the case N=5 and r=2. The results will be compared with each other and with the ones of the direct diagonalization. On the one hand this indicates the unproved completeness of the Bethe ansatz, on the other one some problems occuring in concrete computations can be pointed out.
The original Bethe ansatz will be examined first. The equations (118) and (120) have to be solved for every pair of Bethe quantum numbers κ_1 and κ_2. It will become clear later on that interchanging the quantum numbers does not change the solution so that it suffices to solve the equations for only half of the pairs [25]:

$$0 \leq \kappa_1 \leq \kappa_2 \leq (N-1) \tag{138}$$

These pairs will now be divided up into four classes:

$$\begin{aligned}
C_1 &= \{(\kappa_1,\kappa_2)|\kappa_1 = 0\} \\
C_2 &= \{(\kappa_1,\kappa_2)|\kappa_2-\kappa_1 \geq 2, \kappa_1 \neq 0\} \\
C_3 &= \{(\kappa_1,\kappa_2)|\kappa_2-\kappa_1 = 1, \kappa_1 \neq 0\} \\
C_4 &= \{(\kappa_1,\kappa_2)|\kappa_2-\kappa_1 = 0, \kappa_1 \neq 0\}
\end{aligned} \tag{139}$$

As the number of pairs is greater than the number of eigenvalues $\binom{5}{2}$, apparently either the equations are not solvable for every pair or the solution to some pairs give the same coordinates and eigenvalues. It may also occur that there are several solutions for a given pair of Bethe quantum numbers or that there are solutions associated with wrong eigenvalues. In both of the cases every solution which constitutes a wrong eigenvalue simply gives a zero eigenstate.
It becomes clear immediately that on C_1 the relevant solutions are given analytically by:

$$\begin{aligned} f_1 &= 0 \\ f_2 &= \frac{2\pi\kappa_2}{N} \\ \phi &= 0 \end{aligned} \tag{140}$$

To find the solutions on the remaining classes, the equations have to be solved numerically. The routine available for this paper uses the Newton method, which requires good starting values. Therefore it is helpful to derive an equation for just one of the undeterminates. This can be understood as a function in one variable, which zeros are the solutions. Plotting the function gives a quite good idea where these lie.
Defining

$$k = \frac{2\pi}{N}(\kappa_1 + \kappa_2) = f_1 + f_2, \tag{141}$$

(118) and (120) can be combined to an equation for f_1

$$2\cot\left(\frac{Nf_1}{2}\right) - \cot\left(\frac{f_1}{2}\right) + \cot\left(\frac{k - f_1}{2}\right) = 0, \tag{142}$$

where the values of k are evident from the respective class.
The last two equations show that f_1 and f_2 just depend on k, so that interchanging κ_1 and κ_2 does not affect them. Subtracting the both equations in (120) gives:

$$\phi = \frac{N}{2}(f_1 - f_2) - \pi(\kappa_1 - \kappa_2) \tag{143}$$

This means that interchanging κ_1 and κ_2 gives two Bethe roots ϕ and $\tilde{\phi}$, related to each other by

$$\tilde{\phi} = \phi - 2\pi(\kappa_1 - \kappa_2) \tag{144}$$

so that in both cases the same solution (116) results, justifying (138).
On the class C_2 the relation (142) can be used directly, as it turns out quickly that it suffices to look for real solutions. As an example in (Figure 7) the case $k = 4\pi/5$ is depicted, showing one zero. Plotting the function over a greater range is unnecessary, because it is periodic with the chosen one. On C_3 and C_4

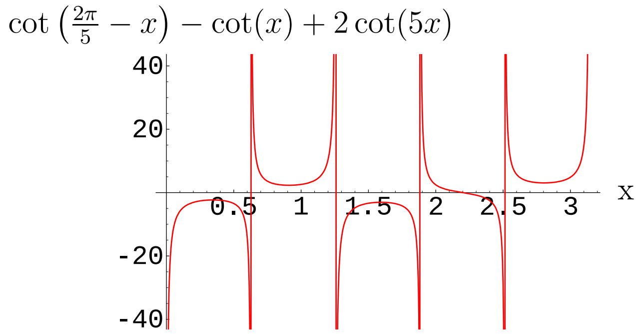

Figure 7: Relation (142) with $f_1/2 = x$ for $k = 4\pi/5$

in addition complex solutions have to be considered. Postulating

$$\begin{aligned} f_1 &= \frac{k}{2} + iv \\ f_2 &= \frac{k}{2} - iv \\ \phi &= \theta + i\chi, \end{aligned} \tag{145}$$

again (118) and (120) can be used to derive an equation for just one of the undeterminates

$$\cos\left(\frac{k}{2}\right)\sinh(Nv) = \sinh([N-1]\,v) + \cos(\theta)\sinh(v), \tag{146}$$

where $\theta = \pi(\kappa_1 - \kappa_2)$ and $\chi = Nv$. This relation can be used the same way as (142).

The solutions are given in (Table 5, 6, 7, 8), where the horizontal lines divide the results due to the four classes. Comparing them with the results of the direct diagonalization, given in (Table 3, 4), shows that the original Bethe ansatz gives the correct eigenvalues. The eigenstates are different, but both of the sets span the correct space. It is interesting that there are no solutions on C_3. Analyzing the relation (146) shows that, for the chosen example, there are only real solutions on C_3. These however do not constitute correct results concerning the eigenvalues and therefore give zero eigenstates, as explained above.

The Bethe ansatz equations in the new set of undeterminates μ_i are given by:

$$\begin{aligned} \left(\frac{\mu_1 - i}{\mu_1}\right)^5 &= \frac{\mu_1 - \mu_2 - i}{\mu_1 - \mu_2 + i} \\ \left(\frac{\lambda_2 - i}{\lambda_2}\right)^5 &= \frac{\mu_2 - \mu_1 - i}{\mu_2 - \mu_1 + i}. \end{aligned} \tag{147}$$

Defining

$$\begin{aligned} x &:= \frac{\mu_1 - i}{\mu_1} \\ y &:= \frac{\mu_2 - i}{\mu_2}, \end{aligned} \tag{148}$$

the system (147) can be solved by considering the following polynomial equation

$$x^5 \left(1 - 2x + x^5\right)^5 + \left(x - 2x^5 + x^6\right)^5 = 0, \tag{149}$$

which can be solved numerically. Again solutions giving wrong eigenvalues with zero eigenstates occur. The correct ones are given in (Table 9). The associated eigenstates are due to (Table 10), where $a(\lambda), b(\lambda)$ and $c(\lambda)$ now refer to the entries of the T matrix, and can be read off from (Table 11, 13, 14). Just like the original one, the algebraic Bethe ansatz gives the correct eigenvalues and a set of eigenstates.

An inaccuracy related to numerics occurs in the context of wrong eigenvalues. In some cases the components of the zero eigenstates are of the same order as some of the non-zero ones. For clarity such states have been multiplied by an appropriate real number.

Nevertheless, in general it can be difficult to distinguish between the correct and the wrong eigenvalues and eigenstates. Concerning concrete computations, this stresses the usefulness of the Bethe ansatz especially for big systems, where a direct diagonalization is not possible anymore.

5. Summary and outlook

In this thesis the algebraic background of integrability concerning classical and quantum mechanical spin systems has been formulated in a compact way. This consists of two different formalisms, both of which are based on the existence of a bialgebra structure together with one additional postulate. The first one is due to Ballesteros and Ragnisco [7] and requires a non-empty center. Originally based on usual higher coproducts, a new extension to so-called pseudo-coproducts has been constructed. These are defined on sets of integers and generalize the standard coproduct, which enables to clarify the algebraic theory behind the $\mathcal{B}$-partitioned spin systems of Schmidt and Steinigeweg [8].

The second formalism takes up an idea of Drinfel´d [9], which makes it possible to construct RTT algebras exclusively based on quasicocommutative bialgebras. This gives a structure completely analogous to the one of Ballesteros and Ragnisco. As far as the author knows, the formalism has been used for the first time to deduce the XXX-S=1/2-Heisenberg chain from a modified Yangian evaluation representation, derived in this thesis. The resulting algebraic Bethe ansatz has been compared to the coordinate Bethe ansatz, which denotes the original calculation of Bethe and Hulthén, and the relation between them has been examined. The Bethe ansatz equations have been solved for an easy model. The results have been compared with each other and with the ones gained from a direct diagonalization.

Due to a lack of time, some immediate questions remain unanswered. As mentioned in the text, the introduced pseudo-coproduct gives the usual higher coproduct corresponding to the standard one if the full N particle situation is considered. Therefore the question arises whether it is possible to generalize this to other coproducts. As Quantum Groups give the most important physical examples, it would be of special interest to examine deformed versions of the standard coproduct. The associated "deformed" versions of the $\mathcal{B}$-partitioned spin systems would be highly interesting, at least from a formal point of view.

Another question is far-reaching and aims at a better understanding of integrability itself. As pointed out in the third section, integrability in the quantum case mainly consists of two methods, which enable to diagonalize the Hamiltonian of a physical system analytically. Both of these are based on algebraic structures, which can be constructed systematically. The relation of these two branches, on the algebraic as well as the methodical level, is more or less completely unknown [7]. Of course a better understanding of the relations would be a big step into the direction of a more general theory of integrability.

The last big question of course refers to the construction of new integrable systems. Concerning the extended approach of Ballesteros and Ragnisco, this could be done based on well-known mathematical structures. In this thesis the formalism was applied to the $\mathrm{sl}(2, \mathbb{C})$, whereas in [7] the original one was used together with other bialgebras to deduce interesting integrable systems. Hence it would

be a natural next step to apply the extended version to those bialgebras and examine the resulting more general versions of the systems.
Regarding Drinfel´d´s formalism, the challenge is of course to use its higher generality. This means that new, physically significant quasicocommutative bialgebras have to be constructed, which are not quasitriangular. Obviously this a tremendous task, which probably will not be solved that fast.

A. Appendix

A.1. Modules and representations

In the following modules and their relation to representations will be discussed briefly [17, 26]:

DEFINITION A-1. *Let $(\mathcal{A}, m)$ be an algebra. A (left $\mathcal{A}$-)module is a pair consisting of a vector space V and a linear map $\delta : \mathcal{A} \otimes V \to V$, compatible with the multiplication in the following sense*

$$\delta \circ (\mathrm{id} \otimes \delta) = \delta \circ (m \otimes \mathrm{id})\,. \tag{150}$$

If the algebra has a unit element, δ has to satisfy in addition

$$\delta(1 \otimes v) = v, \tag{151}$$

where $v \in V$.

Now it can be made connection to representations:

PROPOSITION A-1. *Considering*

$$\rho(X)(v) = \delta(X \otimes v) \tag{152}$$

for $X \in \mathcal{A}$, every module (V, δ) defines a representation ρ and the other way round.

Proof. The proposition follows by direct computation:
"$\Rightarrow$":

$$\begin{aligned}
\rho(XY)(v) &= \delta(XY \otimes v) \\
&= \delta \circ (m \otimes \mathrm{id})(X \otimes Y \otimes v) \\
&= \delta \circ (\mathrm{id} \otimes \delta)(X \otimes Y \otimes v) \\
&= \delta(X \otimes \delta(Y \otimes v)) \\
&= \rho(X)\,(\rho(Y)(v)) = \rho(X)\rho(Y)(v)
\end{aligned}$$

It remains to check the relation associated with (151):

$$\rho(1)(v) = \delta(1 \otimes v) = v \Rightarrow \rho(1) = 1$$

"$\Leftarrow$":

$$\begin{aligned}
\delta \circ (m \otimes \mathrm{id})(X \otimes Y \otimes v) &= \delta(XY \otimes v) \\
&= \rho(XY)(v) \\
&= \rho(X)\rho(Y)(v) \\
&= \rho(X)(\rho(Y)(v)) \\
&= \delta(X \otimes \delta(Y \otimes v)) = \delta \circ (\mathrm{id} \otimes \delta)(X \otimes Y \otimes v)
\end{aligned}$$

Finally (151) has to be checked, completning the proof

$$\delta(1 \otimes v) = \rho(1)(v) = v,$$

which is of course due to the algebra homomorphism property of the representation:

$$\rho(1) = \rho(11) = \rho(1)\rho(1) \Rightarrow \rho(1) = 1$$

□

It is common to use the notion "representation" to refer to the map ρ, as well as the corresponding vector space. Due to Proposition A-1, it is convenient therefore to talk about modules in the context of vector spaces associated with representations.
Often modules are defined slightly more general in terms of rings. Although in most of the literature, important for the subject of this thesis, this is not the case, it is worthwhile to have a look on this alternative definition, because it gives a good idea of what modules are:

DEFINITION A-2. *Let $\mathcal{R}$ be a ring and $\mathcal{M}$ an abelian group. If $\cdot : \mathcal{R} \times \mathcal{M} \to \mathcal{M}$ satisfies*

$$r \cdot (x + y) = r \cdot x + r \cdot y \tag{153}$$

$$(r + s) \cdot x = r \cdot x + s \cdot x \tag{154}$$

$$(rs) \cdot x = r \cdot (s \cdot x) \tag{155}$$

for all $r, s \in \mathcal{R}$ and $x \in \mathcal{M}$, $(\mathcal{M}, \cdot)$ is called a left $\mathcal{R}$-module. If $\mathcal{R}$ has a unit element

$$1 \cdot x = x \tag{156}$$

has to be satisfied.

Obviously every module in the sense of Definition A-1 is a module in the sense of Definition A-2, by setting $\delta = \cdot$. The converse is of course not true.
As every field is a ring, a vector space is nothing but a module over the associated field. Hence modules generalize the notion of a vector space.

A.2. The universal enveloping algebra

A Lie algebra $(\mathcal{L}, \lceil ., . \rceil)$, where $\lceil ., . \rceil$ denotes the Lie bracket, has the important restriction that no product is defined. Hence the question arises if it is possible to construct an algebra which contains $\mathcal{L}$ as a vector space and reproduces the Lie bracket through the commutator. The first step towards this universal enveloping algebra $\mathrm{U}(\mathcal{L})$ is to define the free algebra $\mathrm{F}(\mathcal{L})$:

DEFINITION A-3. *Let* $(\mathcal{L}, \lceil ., . \rceil)$ *be a Lie-algebra and* $\{X_i\}$ *a basis. The free algebra F(*$\mathcal{L}$*) is the set of all formal products*

$$\left\{ \sum_k a^{i_1 \ldots i_k} X_{i_1} \ldots X_{i_k} \right\}. \tag{157}$$

Obviously already the free algebra satisfies the first condition $\mathcal{L} \subset \mathrm{F}(\mathcal{L})$. Therefore it is clear that the universal enveloping algebra results by identifying the Lie bracket with the commutator of the formal product:

DEFINITION A-4. *Let* $(\mathcal{L}, \lceil ., . \rceil)$ *be a Lie algebra with basis* $\{X_i\}$ *and* $\mathcal{I}$ *the ideal of F(*$\mathcal{L}$*), generated by*

$$\{\lceil X_i, X_j \rceil - X_i X_j + X_j X_i\} \tag{158}$$

Then the universal enveloping algebra is defined by the following quotient:

$$U(\mathcal{L}) = F(\mathcal{L})/\mathcal{I} \tag{159}$$

The term "universal" results from the fact that obviously every map from $(\mathcal{L}, \lceil ., . \rceil)$ on an other Lie algebra factorizes over U($\mathcal{L}$). Similar considerations are well-known from the tensor product.
The representation theory of the universal enveloping algebra results immediately from the one of the Lie algebra by extending the maps as algebra homomorphisms.

A.3. The octahedron as a $\mathcal{B}$-partitioned spin system

The labels of the $S = 1/2$-octrahedron, due to Theorem 3-3.

$\|S_N S_N^3 S_{1234} S_{14} S_{23} S_{56}\rangle$	$\|S_N S_N^3 S_{1234} S_{14} S_{23} S_{56}\rangle$
0 0 0 0 0 0	2 (-2) 1 0 1 1
1 (-1) 1 1 0 0	2 (-1) 1 0 1 1
1 0 1 1 0 0	2 0 1 0 1 1
1 1 1 1 0 0	2 1 1 0 1 1
1 (-1) 1 0 1 0	2 2 1 0 1 1
1 0 1 0 1 0	1 (-1) 0 1 1 1
1 1 1 0 1 0	1 0 0 1 1 1
0 0 0 1 1 0	1 1 0 1 1 1
1 (-1) 1 1 1 0	0 0 1 1 1 1
1 0 1 1 1 0	1 (-1) 1 1 1 1
1 1 1 1 1 0	1 0 1 1 1 1
2 (-2) 2 1 1 0	1 1 1 1 1 1
2 (-1) 2 1 1 0	2 (-2) 1 1 1 1
2 0 2 1 1 0	2 (-1) 1 1 1 1
2 1 2 1 1 0	2 0 1 1 1 1
2 2 2 1 1 0	2 1 1 1 1 1
1 (-1) 0 0 0 1	2 2 1 1 1 1
1 0 0 0 0 1	1 (-1) 2 1 1 1
1 1 0 0 0 1	1 0 2 1 1 1
0 0 1 1 0 1	1 1 2 1 1 1
1 (-1) 1 1 0 1	2 (-2) 2 1 1 1
1 0 1 1 0 1	2 (-1) 2 1 1 1
... continued on the next page	

Table 1: Labels of the $S = 1/2$-octrahedron resulting from Theorem 3-3

$\lvert S_N S_N^3 S_{1234} S_{14} S_{23} S_{56}\rangle$	$\lvert S_N S_N^3 S_{1234} S_{14} S_{23} S_{56}\rangle$
1 1 1 1 0 1	2 0 2 1 1 1
2 (-2) 1 1 0 1	2 1 2 1 1 1
2 (-1) 1 1 0 1	2 2 2 1 1 1
2 0 1 1 0 1	3 (-3) 2 1 1 1
2 1 1 1 0 1	3 (-2) 2 1 1 1
2 2 1 1 0 1	3 (-1) 2 1 1 1
0 0 1 0 1 1	3 0 2 1 1 1
1 (-1) 1 0 1 1	3 1 2 1 1 1
1 0 1 0 1 1	3 2 2 1 1 1
1 1 1 0 1 1	3 3 2 1 1 1

Table 2: Labels of the $S = 1/2$-octrahedron resulting from Theorem 3-3

A.4. Numerical results for the XXX-S=1/2-Heisenberg chain

For N=5 the 2-magnon sector of $\mathcal{H}$ is given by:

$$\langle|11000\rangle, |10100\rangle, |10010\rangle, |10001\rangle, |01100\rangle, |01010\rangle, |01001\rangle, |00110\rangle, |00101\rangle, |00011\rangle\rangle$$
$$:= \langle b_1, \ldots, b_{10}\rangle \tag{160}$$

In the following tables the columns labelled by b_i show the component of the respective basis vector.

$E - E_0$	b_1	b_2	b_3	b_4
0	1	1	1	1
1.38197	0	-0.381966	0.618034	1.61803
1.38197	-2.61803	0	-1	-4.23607
1.76393	0	0.381966	0.236068	-0.618034
1.76393	2.61803	0	-1	-1.61803
3.61803	0	-2.61803	-1.61803	-0.618034
3.61803	-0.381966	0	-1	0.236068
4	1	-1	-1	1
6.23607	0	2.61803	-4.23607	1.61803
6.23607	0.381966	0	-1	0.618034

Table 3: Eigenvalues and Eigenvectors resulting from direct diagonalization

b_5	b_6	b_7	b_8	b_9	b_{10}
1	1	1	1	1	1
-1.61803	-0.618034	0.381966	-1	0	1
2.61803	1.61803	-1.61803	4.23607	1	0
0.618034	-0.236068	-0.381966	-1	0	1
-2.61803	-0.618034	0.618034	1.61803	1	0
0.618034	1.61803	2.61803	-1	0	1
0.381966	-0.618034	0.618034	-0.236068	1	0
1	-1	-1	1	-1	1
-1.61803	4.23607	-2.61803	-1	0	1
-0.381966	1.61803	-1.61803	-0.618034	1	0

Table 4: Eigenvalues and Eigenvectors resulting from direct diagonalization

κ_1	κ_2	f_1	f_2	$2\epsilon = E - E_0$
0	0	0	0	0
0	1	0	2 π /5	1.38197
0	2	0	4 π /5	3.61803
0	3	0	6 π /5	3.61803
0	4	0	8 π /5	1.38197
1	3	1.705325	3.321222	6.23607
1	4	1.570796	4.712389	4
2	4	2.961962	4.577859	6.23607
1	1	1.256637 + i 1.198913	1.256637 - i 1.198913	1.76393
4	4	5.026555 + i 1.198913	5.026555 - i 1.198913	1.76393

Table 5: Eigenvalues resulting from the coordinate Bethe ansatz

b_1	b_2	b_3	b_4
2	2	2	2
-0.5 + i 1.538842	-0.5 + i 0.363271	0.618034	1.309017 + i 0.951057
-0.5 - i 0.363271	-0.5 + i 1.538842	-1.618033	0.190983 + i 0.587785
-0.5 + i 0.363271	-0.5 - i 1.538842	-1.618034	0.190983 - i 0.587785
-0.5 - i 1.538842	-0.5 - i 0.363271	0.618034	1.309017 - i 0.951057
-0.217066 - i 0.668057	1.487794 + i 1.080939	-1.839013	0.568286 - i 0.412887
1.414213	-1.414213	-1.414213	1.414213
-0.217066 + i 0.668057	1.487794 - i 1.080939	-1.839013	0.568286 + i 0.412887
-5.020232 - i 3.647419	0.732444 - i 2.254231	2.370239	1.917562 + i 5.901648
-5.020232 + i 3.647419	0.732444 + i 2.254231	2.370239	1.917562 - i 5.901648

Table 6: Eigenvectors resulting from the coordinate Bethe ansatz

b_5	b_6	b_7
2	2	2
-1.618034	-0.5 - i 0.363271	0.190983 + i 0.587785
0.618034	-0.5 - i 1.538842	1.309017 - i 0.951056
0.618034	-0.5 + i 1.538842	1.309017 + i 0.951056
-1,618034	-0.5 + i 0.363271	0.190983 - i 0.587785
-0.702437	1.487787 - i 1.080949	-0.568279 + i 1.749008
1.414213	-1.414213	-1.414213
-0.702437	1.487787 + i 1.080949	-0.568279 - i 1.749008
6.205360	0.732444 + i 2.254231	-1.917563 + i 1.393191
6.205360	0.732444 - i 2.254231	-1.917563 - i 1.393191

Table 7: Eigenvectors resulting from the coordinate Bethe ansatz

b_8	b_9	b_{10}
2	2	2
-0.5 - i 1.538842	0.190983 - i 0.587785	1.309017 - i 0.951057
-0.5 + i 0.363271	1.309017 + i 0.951057	0.190983 - i 0.587785
-0.5 - i 0.363271	1.309012 - i 0.951057	0.190983 + i 0.587785
-0.5 + i 1.538842	0.190983 + i 0.587785	1.309017 + i 0.951057
-0.217062 + i 0.668058	-0.568294 - i 1.749000	0.568286 + i 0.412879
1.414213	-1.414213	1.414213
-0.217062 - i 0.668058	-0.568294 + i 1.749000	0.568286 - i 0.412879
-5.020242 + i 3.647419	-1.912563 - i 1.393191	1.917562 - i 5.901648
-5.020242 + i 3.647419	-1.912563 + i 1.393191	1.917562 + i 5.901648

Table 8: Eigenvectors resulting from the coordinate Bethe ansatz

$E - E_0$	μ_1	μ_2
0	∞	- ∞
1.38197	0.688191 + i/2	i $4.503599 \cdot 10^{15}$
1.38197	- 0.688191 + i/2	i $4.503599 \cdot 10^{15}$
1.76393	- 0.317019 - i 0.002499	-0.317042 +i 1.024985
1.76393	0.317019 - i 0.002499	0.317042 +i 1.024985
3.61803	0.162010 + i/2	i $4.503599 \cdot 10^{15}$
3.61803	-0.162010 + i/2	i $4.503599 \cdot 10^{15}$
4	-1/2 + i/2	1/2 +i/2
6.23607	- 0.436886 + i/2	0.045029 + i/2
6.23607	0.436886 + i/2	0.045029 + i/2

Table 9: Eigenvalues resulting from the algebraic Bethe ansatz

b_1	$a(\lambda_1)^4c(\lambda_1)a(\lambda_2)^4c(\lambda_2) + a(\lambda_1)^3b(\lambda_1)c(\lambda_1)a(\lambda_2)^3b(\lambda_2)c(\lambda_2)$
b_2	$a(\lambda_1)^3b(\lambda_1)c(\lambda_1)a(\lambda_2)^4c(\lambda_2) + a(\lambda_1)^2c(\lambda_1)^3a(\lambda_2)^3b(\lambda_2)c(\lambda_2) + a(\lambda_1)^3b(\lambda_1)c(\lambda_1)a(\lambda_2)^2b(\lambda_2)^2c(\lambda_2)$
b_3	$a(\lambda_1)^3b(\lambda_1)c(\lambda_1)a(\lambda_2)^3b(\lambda_2)c(\lambda_2) + a(\lambda_1)^2b(\lambda_1)^2c(\lambda_1)a(\lambda_2)^2b(\lambda_2)^2c(\lambda_2)$
b_4	$a(\lambda_1)^2b(\lambda_1)^2c(\lambda_1)a(\lambda_2)^4c(\lambda_2) + a(\lambda_1)b(\lambda_1)c(\lambda_1)^3a(\lambda_2)^3b(\lambda_2)c(\lambda_2) + a(\lambda_1)^2c(\lambda_1)^3a(\lambda_2)^2b(\lambda_2)^2c(\lambda_2) + a(\lambda_1)^3b(\lambda_1)c(\lambda_1)a(\lambda_2)b(\lambda_2)^3c(\lambda_2)$
b_5	$a(\lambda_1)^2b(\lambda_1)^2c(\lambda_1)a(\lambda_2)^3b(\lambda_2)c(\lambda_2) + a(\lambda_1)b(\lambda_1)c(\lambda_1)^3a(\lambda_2)^2b(\lambda_2)^2c(\lambda_2) + a(\lambda_1)^2b(\lambda_1)^2c(\lambda_1)a(\lambda_2)b(\lambda_2)^3c(\lambda_2)$
b_6	$a(\lambda_1)^2b(\lambda_1)^2c(\lambda_1)a(\lambda_2)^2b(\lambda_2)^2c(\lambda_2) + a(\lambda_1)b(\lambda_1)^3c(\lambda_1)a(\lambda_2)b(\lambda_2)^3c(\lambda_2)$
b_7	$a(\lambda_1)b(\lambda_1)^3c(\lambda_1)a(\lambda_2)^4c(\lambda_2) + b(\lambda_1)^2c(\lambda_1)^3a(\lambda_2)^3b(\lambda_2)c(\lambda_2) + a(\lambda_1)c(\lambda_1)^3b(\lambda_1)a(\lambda_2)^2b(\lambda_2)^2c(\lambda_2) + a(\lambda_1)^2c(\lambda_1)^3a(\lambda_2)b(\lambda_2)^3c(\lambda_2) + a(\lambda_1)^3b(\lambda_1)c(\lambda_1)b(\lambda_2)^4c(\lambda_2)$
b_8	$a(\lambda_1)b(\lambda_1)^3c(\lambda_1)a(\lambda_2)^3b(\lambda_2)c(\lambda_2) + b(\lambda_1)^2c(\lambda_1)^3a(\lambda_2)^2b(\lambda_2)^2c(\lambda_2) + a(\lambda_1)b(\lambda_1)c(\lambda_1)^3a(\lambda_2)b(\lambda_2)^3c(\lambda_2) + a(\lambda_1)^2b(\lambda_1)^2c(\lambda_1)b(\lambda_2)^4c(\lambda_2)$
b_9	$a(\lambda_1)b(\lambda_1)^3c(\lambda_1)a(\lambda_2)^2b(\lambda_2)^2c(\lambda_2) + b(\lambda_1)^2c(\lambda_1)^3a(\lambda_2)b(\lambda_2)^3c(\lambda_2) + a(\lambda_1)b(\lambda_1)^3c(\lambda_1)b(\lambda_2)^4c(\lambda_2)$
b_{10}	$a(\lambda_1)b(\lambda_1)^3)c(\lambda_1)a(\lambda_2)b(\lambda_2)^3c(\lambda_2) + b(\lambda_1)^4c(\lambda_1)b(\lambda_2)^4c(\lambda_2)$

Table 10: Eigenvectors resulting from the algebraic Bethe ansatz

b_1	b_2	b_3
2	2	2
1.076997 + i3.314655	-4.113761 + i 1.266086	-2.819614 + i 2.048569
1.076997 - i3.314655	-4.113761 - i 1.266086	-2.819614 - i 2.048569
6.172839 + i4.484830	2.357814 - i1.7130527	-2.357815 - i 7.256608
6.172839 - i4.484830	2.357814 + i1.7130527	-2.357815 + i 7.256608
1.571317 - i1.141629	-4.113761 + i2.9888226	-0.600189 + i 1.847194
1.571317 + i1.141629	-4.113761 - i2.9888226	-0.600189 - i 1.847194
4	-4	4
0.6172839 + i1.8998047	1.616070 - i4.9737532	-1.616070 + i 1.174144
0.6172839 - i1.8998047	1.616070 + i4.9737532	-1.616070 - i 1.174144

Table 11: Eigenvectors resulting from the algebraic Bethe ansatz

b_4	b_5	b_6
2	2	2
1.076997 - i 0.782484	-1.331241 + i 3.425394 10^{-16}	-2.819614 - i 2.048569
1.076997 + i 0.782484	-1.331241 - i 3.425394 10^{-16}	-2.819614 + i 2.048569
-0.900605 - i 2.771778	-2.914419 - i 6.938894 10^{-15}	-2.357815 + i 7.256608
-0.900605 + i 2.771778	-2.914419 + i 6.938894 10^{-15}	-2.357815 - i 7.256608
1.571317 - i 4.836017	5.084889 - i 2.854495 10^{-15}	-0.600189 - i 1.847194
1.571317 + i 4.836017	5.084889 + i 2.854495 10^{-15}	-0.600189 + i 1.847194
-4	-4	4
-4.230927 + i 3.073985	5.229714 + i 3.903128 10^{-16}	-1.616070 - i 1.174144
-4.230927 - i 3.073985	5.229714 - i 3.903128 10^{-16}	-1.616070 + i 1.174144

Table 12: Eigenvectors resulting from the algebraic Bethe ansatz

b_7	b_8
2	2
3.485235 + i 4.567193 10^{-16}	1.076997 + i 0.782484
3.485235 - i 4.567193 10^{-16}	1.076997 - i 0.782484
-7.630049 - i 1.110223 10^{-13}	-0.900605+ i 2.771778
-7.630049 + i 1.110223 10^{-13}	-0.900605- i 2.771778
-1.942255 - i 1.141798 10^{-15}	1.571317+ i 4.836017
-1.942255 + i 1.141798 10^{-15}	1.571317- i 4.836017
4	-4
1.997573 + i 9.020562 10^{-15}	-4.230927- i 3.073948
1.997573 - i 9.020562 10^{-15}	-4.230927+ i 3.073948

Table 13: Eigenvectors resulting from the algebraic Bethe ansatz

b_9	b_{10}
2	2
-0.411376 -i 1.266086	1.076997 - i 3.314655
-0.411376 +i 1.266086	1.076997 + i 3.314655
2.357815 + i 1.713053	6.172839 - i 4.484830
2.357815 - i 1.713053	6.172839 + i 4.484830
-4.113861 +i 2.988823	1.571317 + i 1.141629
-4.113861 -i 2.988823	1.571317 - i 1.141629
-4	4
1.616070 +i 4.973753	0.617284 - i 1.899804
1.616070 -i 4.973753	0.617284 + i 1.899804

Table 14: Eigenvectors resulting from the algebraic Bethe ansatz

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Acknowledgements

This diploma thesis was written in the group "Makroskopische Systeme und Quantentheorie", supervised by apl. Prof. Dr. Heinz-Jürgen Schmidt. I am very grateful to him for his outstanding support, not only during the working process, but also in the time before. I attended two of his reading courses dealing with mathematical concepts in physics, which confirmed me in my decision to work on theoretical physics and were a good introduction to this thesis.
I want to thank Jun.-Prof. Dr. Jochen Gemmer for reviewing my thesis. Apl. Prof. Dr. Heinz-Jürgen Schmidt, Prof. Dr. Gunnar Borstel and apl. Prof. Dr. Manfred Wöhlecke gave me vocational guidance and supported the related applications, for which I want to express my gratitude.
I want to thank Marcel Ogiewa and Kay-Michael Voit for the teamwork during all the lectures and some funny hours. Besides I am grateful to Christian Neupert for interesting discussions and to Jack for never biting me.
The biggest dept of gratitude I owe my parents. They made it possible for me to study physics and always supported me in every respect! This diploma thesis is dedicated to them.

Erklärung

Hiermit erkläre ich, die vorliegende Diplomarbeit selbständig und nur mit den aufgeführten Hilfsmitteln verfasst zu haben.

Osnabrueck, im April 2008

- Björn Erbe -